Group Manifold Approach to Supergravity

超引力的群流形方法

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Abstract

摘要

We present a short review of the group-geometric approach to supergravity theories, from the point of view of recent developments. The central idea is the unification of usual diffeomorphisms, gauge symmetries, and supersymmetries into superdiffeomorphisms in a supergroup manifold. The example of $N = 1$ supergravity in $d = 4$ is discussed in detail and used to illustrate all the steps in the construction of a group manifold action. In the Appendices, we summarize basic notions of group manifold geometry and of integration on supermanifolds.

我们基于最新进展，对超引力理论的群几何方法做简要综述。其核心思想是将通常的微分同胚、规范对称性与超对称性统一为超群流形上的超微分同胚。我们详细讨论了 $N = 1$ 超引力在 $d = 4$ 中的例子，并用它阐明了群流形作用量构造的全部步骤。在附录中，我们总结了群流形几何与超流形积分的基本概念。

Keywords

关键词

Super gravity - Group geometry - Supermanifolds

超引力——群几何——超流形

Introduction

引言

Fundamental interactions are described by field theories with local invariances: the actions that govern their dynamics are invariant under field transformations involving parameters that are (arbitrary) functions of spacetime. This holds true both for gravity and gauge theories, where the local symmetries are general coordinate and gauge transformations, respectively.

基本相互作用由具有定域不变性的场论描述: 支配其动力学的作用量在含任意时空函数参数的场变换下保持不变。该结论对引力和规范理论均成立，二者的定域对称性分别是广义坐标变换和规范变换。

The essential difference between these two types of local transformations, in their infinitesimal versions, is that variations under diffeomorphisms always contain a derivative of the field, which is absent in gauge transformations. As well known, this is due to the fact that general coordinate transformations relate fields at different spacetime points, whereas gauge transformations relate fields at the same spacetime point.

这两类定域变换在无穷小形式下的核心区别在于: 微分同胚的变分始终包含场的导数，而规范变换中不存在这一项。众所周知，这一差异源于广义坐标变换关联不同时空点的场，而规范变换仅关联同一时空点的场。

Nonetheless, it is possible to give a unified description of diffeomorphisms and gauge transformations. This we achieve in a group-geometric framework.

尽管如此，我们仍可对微分同胚和规范变换给出统一描述，本文在群几何框架下实现了这一目标。

The main idea is to consider as basic fields of the theory the components of the vielbein one-form $\sigma^A = \sigma(z)^A_A dz^A$ on the manifold of a Lie (super)group G, A being an index in the G Lie (super)algebra and z^Λ the coordinates of the group manifold. This vielbein satisfies the Maurer-Cartan (MC) equations

核心思路是: 将理论的基本场取为李 (超) 群 G, A 流形上 vielbein 一元形式 $\sigma^A = \sigma(z)^A_A dz^A$ 的分量，其中 G 对应李 (超) 代数指标，z^A 是群流形的坐标。该 vielbein 满足莫雷-嘉当 (MC) 方程:

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \tag{1}$$

where C^A_{BC} are the structure constants of the G Lie algebra. A brief account of group manifold geometry is given in Appendix "Appendix: Group Manifold Geometry".

其中 C^A_{BC} 是 G 李代数的结构常数。附录《附录: 群流形几何》给出了群流形几何的简要介绍。

The G vielbein $\sigma^A\,(z)$ has a fixed dependence on the coordinates z , and therefore cannot be a dynamical object. We must consider a "soft" group manifold, diffeomorphic to G and denoted by $\widetilde{G}$, with a vielbein μ^A not satisfying anymore the MC equations. The amount of deformation from the original "rigid" group manifold is measured by the curvature two-form:

原 vielbein $\sigma^A\,(z)$ 对坐标 z 具有固定依赖关系，因此不是动力学对象。我们需要考虑一个“软”群流形，它与 G 微分同胚，记为 $\widetilde{G}$，其 vielbein μ^A 不再满足 MC 方程。相对于原始“刚性”群流形的形变量由曲率二元形式度量:

$$R^A \equiv d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C \tag{2}$$

Tangent vectors on $\widetilde{G}$, dual to the vielbein μ^A , are denoted by t_B , so that $\mu^A\,(t_B) = \ \delta^A_B$.

$\widetilde{G}$ 上对偶于 vielbein μ^A 的切向量记为 t_B，满足 $\mu^A\,(t_B) = \ \delta^A_B$ 。

Diffeomorphisms along tangent vectors $\varepsilon = \varepsilon^A t_A$ on $\widetilde{G}$ are generated by the Lie derivative ℓ_ε . When applied to the $\widetilde{G}$ vielbein, the variation under diffeomorphisms takes the form:

$\widetilde{G}$ 上沿切向量 $\varepsilon = \varepsilon^A t_A$ 的微分同胚由李导数 ℓ_ε 生成。将其作用于 $\widetilde{G}$ 的 vielbein，微分同胚下的变分形式为:

$$\ell_\varepsilon \mu^A = d\varepsilon^A + C^A_{BC} \mu^B \varepsilon^C + \iota_\varepsilon R^A \tag{3}$$

where ι_ε is the contraction operator (see Appendix "Appendix: Group Manifold Geometry"). On the right-hand side, one recognizes the G -covariant derivative of the infinitesimal parameter ε^A plus a curvature term. When the curvature term vanishes, i.e., when $\iota_\varepsilon R^A = 0$, the diffeomorphism takes the form of a gauge transformation, and the curvature is said to be horizontal along the t_A s entering the sum in $\varepsilon = \varepsilon^A t_A$. Thus, in group manifold geometry, gauge transformations can be interpreted as particular diffeomorphisms, along the directions on which the curvatures are horizontal.

其中 ι_ε 是缩并算子 (参见附录《附录: 群流形几何》)。在等式右侧可以看出，它是无穷小参数 ε^A 的 G 协变导数加上曲率项。当曲率项为零，即 $\iota_\varepsilon R^A = 0$ 时，微分同胚具有规范变换的形式，此时称曲率沿 $\varepsilon = \varepsilon^A t_A$ 求和项中的 t_A s 方向是水平的。因此，在群流形几何中，规范变换可以解释为沿曲率水平方向的特殊微分同胚。

This group-geometric setting is particularly suited to supergravity theories, where local supersymmetry variations can be interpreted as diffeomorphisms in the superPoincaré group manifold, along the fermionic directions. It is then clear how to proceed to find theories invariant under local supersymmetry transformations: we must devise a procedure that yields actions, invariant under superdiffeomorphisms. This is very similar in spirit to the superspace approach [1, 2] , where supergravity multiplets of dynamical (and auxiliary) fields are contained into a single superfield depending on superspace coordinates, or to the gauge supersymmetry framework, based on the superPoincaré group, initiated in [3]. However, the group manifold approach has important differences, as we explain in the coming sections.

> 这种群几何框架特别适用于超引力理论: 定域超对称变换可以解释为超庞加莱群流形上沿费米方向的微分同胚。寻找定域超对称变换下不变的理论的思路也十分清晰: 我们需要设计一套流程，得到超微分同胚下不变的作用量。这一思路在精神上与超空间方法 [1, 2] 非常相似——超引力的动力学 (及辅助) 场多重态都包含在依赖于超空间坐标的单个超场中；它也类似于由文献 [3] 开创、基于超庞加莱群的规范超对称框架。但正如我们后续章节将要说明的，群流形方法存在重要差异。

The action is obtained with an algorithmic procedure, as the integral of a d - form,"living" on the whole supergroup (soft) manifold $\widetilde{G}$, but integrated on a d -dimensional bosonic submanifold of $\widetilde{G}$. This leads to an ordinary spacetime action containing the dynamical fields (and possibly also the auxiliary fields) of a d -dimensional supergravity theory. This algorithm will be discussed in detail and applied to obtain the action of $N = 1, d = 4$ supergravity.

> 该作用量通过算法流程得到，是定义在整个超群 (软) 流形 $\widetilde{G}$ 上的 d -形式的积分，但积分区域是 $\widetilde{G}$ 的一个 d 维玻色子子流形。由此得到的普通时空作用量包含 d 维超引力理论的动力学场 (还可能包含辅助场)。本文将详细讨论该算法，并将其应用于推导 $N = 1, d = 4$ 超引力的作用量。

The original references, where this approach was first proposed, are given in [4-7]. Reviews can be found in [8-13], and [14] is a standard reference for the use of differential forms in gravity and gauge theories.

> 该方法最初提出的原始参考文献见 [4-7]，相关综述可见 [8-13]，其中 [14] 是微分形式应用于引力和规范理论的标准参考文献。

The paper is organized as follows: In section "Supergravity from SuperPoincaré Geometry," we recall the algebraic basis of $d = 4, N = 1$ supergravity as a theory on the (soft) superPoincaré manifold, and the passage to a spacetime action. Section "Symmetries" deals with the symmetries of the spacetime action, as inherited from the diffeomorphism invariances of the group manifold action. In section "Variational Principle and Field Equations," the variational principle is formulated for the group manifold action, and equations of motion are derived. The building rules for (super)group manifold actions are discussed in section "Building Rules" and applied to arrive unambiguously at the group manifold Lagrangian for $N = 1, d = 4$ supergravity. Some conclusions and a selected list of applications and advantages of the group-geometric approach are discussed in section "Conclusions." Finally, the Appendices contain brief accounts of group manifold geometry, integration on supermanifolds and gamma matrix properties.

本文结构安排如下: 在“超庞加莱几何导出超引力”一节，我们回顾了 $d=4, N=1$ 超引力作为 (软) 超庞加莱流形上理论的代数基础，以及推导时空作用量的过程。“对称性”一节讨论时空作用量继承自群流形作用量微分同胚不变性的对称性。在“变分原理与场方程”一节，我们为群流形作用量建立了变分原理并推导了运动方程。(超) 群流形作用量的构造规则在“构造规则”一节讨论，并被用于明确推导得到 $N=1, d=4$ 超引力的群流形拉格朗日量。我们在“结论”一节讨论了相关结论，列出了群几何方法的部分应用与优势。最后，附录简要介绍了群流形几何、超流形积分以及伽马矩阵性质。

Supergravity from SuperPoincaré Geometry

来自超庞加莱几何的超引力

Soft SuperPoincaré Manifold

软超庞加莱流形

Supergravity in first-order vierbein formalism can be recast in a supergroup-geometric setting as follows: Consider $G=$ superPoincaré group, and denote the vielbein on the $\widetilde{G}$ manifold as $\mu^A=\left(V^a, \omega^{ab}, \psi^\alpha\right)$. The index $A=(a, ab, \alpha)$ runs over the translations P_a , Lorentz rotations M_{ab} , and supersymmetry charges $\bar{Q}_\alpha$ of the superPoincaré Lie algebra:

一阶标架形式的超引力可以按如下方式重构为超群几何框架: 考虑 $G=$ 超庞加莱群，将 $\widetilde{G}$ 流形上的标架记为 $\mu^A=\left(V^a, \omega^{ab}, \psi^\alpha\right)$。指标 $A=(a, ab, \alpha)$ 遍历超庞加莱李代数的平移变换 P_a 、洛伦兹转动 M_{ab} 和超对称荷 $\bar{Q}_\alpha$:

$$[P_a, P_b]=0 \tag{4}$$

$$[M_{ab}, M_{cd}]=-\frac{1}{2}\left(\eta_{ad} M_{bc}+\eta_{bc} M_{ad}-\eta_{ac} M_{bd}-\eta_{bd} M_{ac}\right) \tag{5}$$

$$[M_{ab}, P_c]=-\frac{1}{2}\left(\eta_{bc} P_a-\eta_{ac} P_b\right) \tag{6}$$

$$\left[P_a, \bar{Q}_\alpha\right]=0 \tag{7}$$

$$\left[M_{ab}, \bar{Q}_\beta\right]=-\frac{1}{4} \bar{Q}_\alpha\left(\gamma_{ab}\right)_\beta^\alpha \tag{8}$$

$$\left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\}=-i\left(C \gamma^a\right)_{\alpha\beta} P_a, \tag{9}$$

η being the flat Minkowski metric and $C_{\alpha\beta}$ the charge conjugation matrix. The spinorial generator $\bar{Q}_\alpha \equiv Q^\beta C_{\beta\alpha}$ is a Majorana spinor, i.e., $Q^\beta C_{\beta\alpha}=Q_\beta^{\dagger}\left(\gamma_0\right)^\beta{}_\alpha$. Thus, the superPoincaré manifold has ten bosonic directions with coordinates x^a , y^{ab} , parametrizing translations and Lorentz rotations, and four fermionic directions with Grassmann coordinates θ^α , corresponding to the four supercharges $\bar{Q}_\alpha, \alpha=1, \ldots 4$.

η 是平坦闵可夫斯基度规，$C_{\alpha\beta}$ 是电荷共轭矩阵。旋量生成元 $\bar{Q}_\alpha \equiv Q^\beta C_{\beta\alpha}$ 是马约拉纳旋量，即 $Q^\beta C_{\beta\alpha} = Q^\dagger_\beta (\gamma_0)^\beta{}_\alpha$。因此，超庞加莱流形具有 10 个玻色方向，其坐标为 x^a 、y^{ab} ，参数化平移与洛伦兹转动；另有 4 个费米方向，对应格拉斯曼坐标 θ^α ，对应 4 个超荷 $\bar{Q}_\alpha, \alpha = 1, ..., 4$。

The components of the supervielbein of the $\widetilde{G}$ = (soft) superPoincaré manifold are the vierbein V^a , the spin connection ω^{ab} , and the gravitino ψ^α , corresponding, respectively, to the generators P_a, M_{ab} , and $\bar{Q}_\alpha$.

$\widetilde{G}$ = (软) 超庞加莱流形的超标架分量分别是标架 V^a 、自旋联络 ω^{ab} 和引力微子 ψ^α ，分别对应生成元 P_a, M_{ab} 和 $\bar{Q}_\alpha$ 。

Using the structure constants of the Lie superalgebra, the curvature (2) becomes

利用李超代数的结构常数，曲率 (2) 可写为

$$R^a = dV^a - \omega^a_c V^c - \frac{i}{2}\bar{\psi}\gamma^a\psi \equiv DV^a - \frac{i}{2}\bar{\psi}\gamma^a\psi \tag{10}$$

$$R^{ab} = d\omega^{ab} - \omega^a_c\omega^{cb} \tag{11}$$

$$\rho = d\psi - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi \equiv D\psi \tag{12}$$

defining, respectively, the supertorsion, the Lorentz curvature, and the gravitino field strength. D is the Lorentz covariant exterior derivative. Wedge products between forms are understood when omitted.

分别定义了超挠率、洛伦兹曲率和引力微子场强。D 是洛伦兹协变外导数。省略 wedge 积时默认形式间存在 wedge 积。

Taking the exterior derivative of these definitions yields the Bianchi identities:

对这些定义取外导数可得到比安基恒等式:

$$dR^a - \omega^a_{\ b}R^b + R^a_{\ b}V^b - i\bar{\psi}\gamma^a\rho \equiv DR^a + R^a_{\ b}V^b - i\bar{\psi}\gamma^a\rho = 0 \tag{13}$$

$$dR^{ab} - \omega^a_c R^{cb} + \omega^b_c R^{ca} \equiv DR^{ab} = 0 \tag{14}$$

$$d\rho - \frac{1}{4}\omega^{ab}\gamma_{ab}\rho + \frac{1}{4}R^{ab}\gamma_{ab}\psi \equiv D\rho + \frac{1}{4}R^{ab}\gamma_{ab}\psi = 0 \tag{15}$$

At this stage, all the fields depend on all $\widetilde{G}$ manifold coordinates, corresponding to the generators of the Lie superalgebra: thus, $V^a = V^a(x, y, \theta), \omega^{ab} = \omega^{ab}(x, y, \theta)$, and $\psi^\alpha(x, y, \theta)$, where the coordinates x^a , corresponding to the translations P_a , describe usual spacetime. Moreover, the one-forms V^a, ω^{ab} , and ψ live on the whole $\widetilde{G}$ and therefore can be expanded as

在此阶段，所有场都依赖于对应李超代数生成元的全部 $\widetilde{G}$ 流形坐标: 因此有 $V^a = V^a(x, y, \theta), \omega^{ab} = \omega^{ab}(x, y, \theta)$ 和 $\psi^\alpha(x, y, \theta)$ ，其中对应平移变换 P_a 的坐标 x^a 描述普通时空。此外，一元形式 V^a, ω^{ab} 和 ψ 定义在整个 $\widetilde{G}$ 上，因此可以展开为

$$V^a = V^a_\mu(x,y,\theta)\,dx^\mu + V^a_{\mu\upsilon}(x,y,\theta)\,dy^{\mu\upsilon} + V^a_\alpha(x,y,\theta)\,d\theta^\alpha \tag{16}$$

$$\omega^{ab} = \omega^{ab}_\mu(x,y,\theta)\,dx^\mu + \omega^{ab}_{\mu\upsilon}(x,y,\theta)\,dy^{\mu\upsilon} + \omega^{ab}_\alpha(x,y,\theta)\,d\theta^\alpha \tag{17}$$

$$\psi^\alpha = \psi^\alpha_\mu(x,y,\theta)\,dx^\mu + \psi^\alpha_{\mu\upsilon}(x,y,\theta)\,dy^{\mu\upsilon} + \psi^\alpha_\beta(x,y,\theta)\,d\theta^\beta \tag{18}$$

Group Manifold Action

群流形作用量

The overabundance of field components and their dependence on y and θ coordinates can be tamed by defining an appropriate action principle. To end up with a geometrical theory in four spacetime dimensions, we first construct a four-form Lagrangian L made out of the $\widetilde{G}$ vielbein μ^A and its curvature R^A , according to a few building rules to be discussed in section ”Building Rules”. The resulting Lagrangian for superPoincaré supergravity is given by

场分量过多以及它们对 y 和 θ 坐标的依赖可以通过定义合适的作用量原理来解决。为了最终得到四维时空下的几何理论，我们首先根据“构造规则”一节将要讨论的几条构造规则，由 $\widetilde{G}$ 标架 μ^A 及其曲率 R^A 构造四形式拉格朗日量 L 。超庞加莱超引力的最终拉格朗日量为

$$L = R^{ab}V^cV^d\varepsilon_{abcd} + 4\overline{\psi}\gamma_5\gamma_a\rho V^a \tag{19}$$

We then define an action by integrating this Lagrangian on a four-dimensional submanifold M^4 of the $\widetilde{G}$ manifold, spanned by the x coordinates.

然后我们通过将该拉格朗日量在由 x 坐标张成的 $\widetilde{G}$ 流形的四维子流形 M^4 上积分来定义作用量。

Integration on submanifolds M^d of a d -form L that lives on a g -dimensional bigger space $\widetilde{G}$ is carried out as follows: we multiply L by the Poincaré dual of M^d , a (singular) closed $(g-d)$ -form η_{M^d} that localizes the Lagrangian on the submanifold M^d , and integrate the resulting g -form on the whole $\widetilde{G}$. Thus, the group manifold action has the general expression

定义在 g 维更大空间 $\widetilde{G}$ 上的 d 形式 L 在子流形 M^d 上的积分按如下方式进行: 将 L 乘以 M^d 的庞加莱对偶，即一个将拉格朗日量局域在子流形 M^d 上的 (奇异) 闭 $(g-d)$ 形式 η_{M^d} ，再将得到的 g 形式在整个 $\widetilde{G}$ 上积分。因此，群流形作用量的一般表达式为

$$S = \int_{\widetilde{G}} L \wedge \eta_{M^d} \tag{20}$$

The fields of the theory are those contained in L , i.e., the $\widetilde{G}$ vielbein components, and the embedding functions that define the M^d submanifold of $\widetilde{G}$, present in η_{M^d} . We will see later that the embedding functions do not enter the field equations obtained from the variation of (20). This program makes use of standard integration theory when $\widetilde{G}$ is a bosonic space, but requires some new ingredients when $\widetilde{G}$ is a supermanifold, discussed in Appendix ”Integration on Supermanifolds: Integral Forms.”

该理论的场都包含在 L 中，即 $\widetilde{G}$ 标架分量，以及定义 $\widetilde{G}$ 的 M^d 子流形的嵌入函数，这些都出现在 η_{M^d} 中。我们之后会看到，嵌入函数不会出现在对 (20) 变分得到的场方程中。当 $\widetilde{G}$ 是玻色空间时，这套方案使用标准积分理论；而当 $\widetilde{G}$ 是超流形时，需要一些新的内容，附录“超流形上的积分: 积分形式”对此进行了讨论。

In our $d=4$ supergravity example, the group manifold action is the integral on the 14-dimensional $\widetilde{G}$ = soft superPoincaré manifold:

在我们的 $d=4$ 超引力例子中，群流形作用量是在 14 维 $\widetilde{G}$ = 软超庞加莱流形上的积分:

$$S=\int_{\widetilde{G}}\left(R^{ab}V^cV^d\varepsilon_{abcd}+4\overline{\psi}\gamma_5\gamma_a\rho V^a\right)\eta_{M^4} \tag{21}$$

Spacetime Action

时空作用量

A spacetime action, i.e., an action that is the integral on M^4 of a Lagrangian containing fields depending only on x , is obtained from (21) with a particular choice of η_{M^4} . Integration on y and θ coordinates produces then the spacetime action. This particular Poincaré dual is the product of two pieces, $\eta_{M^4}=\eta_y\wedge\eta_\theta$, where η_y is a (singular) six-form that localizes the Lagrangian on the $y=0$ hypersurface:

时空作用量，即对 M^4 积分、拉格朗日量所含场仅依赖于 x 的作用量，可通过对 η_{M^4} 做特定选取从式 (21) 得到。对 y 和 θ 坐标积分后就得到该时空作用量。这个特殊庞加莱对偶是两部分的乘积 $\eta_{M^4}=\eta_y\wedge\eta_\theta$ ，其中 η_y 是将拉格朗日量局域在 $y=0$ 超曲面上的 (奇异) 六形式:

$$\eta_y=\delta\left(y^{01}\right)\delta\left(y^{02}\right)\cdots\delta\left(y^{23}\right)dy^{01}\wedge dy^{02}\wedge\cdots\wedge dy^{23} \tag{22}$$

Integration on the y coordinates reduces (21) to an integral on the superspace $M^{4|4}$ spanned by x and θ . The y dependence of all fields in L disappears because of the delta functions in η_y , and the ”legs” of L along dy differentials are killed by the product of all independent $dy^{\mu\nu}$ in η . The other piece of η_{M^4} (see Appendix ”Integration on Supermanifolds: Integral Forms”), after integration on θ coordinates, produces an integral on M^4 of a Lagrangian four-form, not depending any more on the θ and on the $d\theta$ differentials. Thus, the action

对 y 坐标积分将式 (21) 约化为对由 x 和 θ 张成的超空间 $M^{4|4}$ 的积分。由于 η_y 中的 δ 函数，L 中所有场对 y 的依赖都消失了，且 L 沿 dy 微分的“分支”被 η 中所有独立的 $dy^{\mu\nu}$ 的乘积消去。η_{M^4} 的另一部分 (见附录“超流形积分: 积分形式”) 在对 θ 坐标积分后，得到对 M^4 的拉格朗日四形式积分，该四形式不再依赖 θ 和 $d\theta$ 微分。因此，作用量

$$\begin{aligned}S_{\text{spacetime}}&=\int_{\widetilde{G}}L\wedge\eta_{M^4}=\int_{M^4}L_{y=0,dy=0,\theta=0,d\theta=0}\\&=\int_{M^4}R^{ab}V^cV^d\varepsilon_{abcd}+4\overline{\psi}\gamma_5\gamma_a\rho V^a\end{aligned} \tag{23}$$

contains only the usual fields $V_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ and $\psi_\mu^\alpha(x)$ of $N = 1$ supergravity and reproduces the first-order supergravity action.

仅包含 $N = 1$ 超引力的常规场 $V_\mu^a(x)$、$\omega_\mu^{ab}(x)$ 和 $\psi_\mu^\alpha(x)$，并重现了一阶超引力作用量。

Note 1: η_y is closed (because it contains ”functions” depending on y multiplied by all the dy differentials) and not exact (because of the Dirac deltas $\delta(y)$) and thus belongs to a nontrivial de Rahm cohomology class. In general, deformations of the M^4 surface generated by diffeomorphisms leave the Poincaré dual η in the same cohomology class, since the Lie derivative commutes with the exterior derivative.

注 1: η_y 是闭的 (因为它包含依赖 y 的“函数”乘以所有 dy 微分) 且不是恰当的 (因为狄拉克 δ 函数 $\delta(y)$)，因此它属于一个非平凡德拉姆上同调类。一般而言，微分同胚生成的 M^4 曲面的形变会让庞加莱对偶 η 保持在同一个上同调类，因为李导数与外导数对易。

Note 2: We will always assume that integration on the Lorentz coordinates has been carried out, so that all fields depend only on x and θ coordinates. Moreover, all curvatures are taken to be horizontal in the Lorentz directions. As a consequence, the theory lives in a superspace $M^{4|4}$ spanned by four bosonic coordinates x^a and four fermionic coordinates θ^α .

注 2: 我们始终假设已经完成了对洛伦兹坐标的积分，因此所有场仅依赖于 x 和 θ 坐标。此外，所有曲率都取为洛伦兹方向上的水平曲率。由此，该理论定义在由四个玻色坐标 x^a 和四个费米坐标 θ^α 张成的超空间 $M^{4|4}$ 中。

Note 3: The spacetime action (23) and its invariance under the supersymmetry transformations (45)-(47) were first found in Ref. [15] in second-order formalism and in [16] in first-order formalism (see also the standard references [17,18] on supergravity).

注 3: 时空作用量 (23) 及其在超对称变换 (45)-(47) 下的不变性，最早由文献 [15] 在二阶形式框架下得到，文献 [16] 在一阶形式框架下得到 (另见超引力的经典参考文献 [17,18])。

Symmetries

对称性

The action (21) is the integral on $\widetilde{G}$ of a top form: it is clearly invariant under diffeomorphisms on $\widetilde{G}$. But what we are really interested in are the symmetries of the spacetime action as given in (23), where the variations are carried out only in the x -dependent fields in $L|_{y=dy=0,\theta=d\theta=0}$. The only symmetries guaranteed a priori are the four-dimensional spacetime diffeomorphisms, the spacetime action being an integral of a four-form on M^4 .

作用量 (21) 是 $\widetilde{G}$ 上最高次形式的积分: 它显然在 $\widetilde{G}$ 的微分同胚下不变。但我们真正关注的是式 (23) 给出的时空作用量的对称性，其中仅对 $L|_{y=dy=0,\theta=d\theta=0}$ 中依赖 x 的场做变分。预先能保证的对称性只有四维时空微分同胚，因为时空作用量是 M^4 上四形式的积分。

Here resides most of the power of the group manifold formalism: if one considers the ”mother” action (20) on $\widetilde{G}$, the guaranteed symmetries are all the diffeomorphisms on $\widetilde{G}$, generated by the Lie derivative ℓ_ε along the tangent vectors $\varepsilon = \varepsilon^A t_A$ of $\widetilde{G}$. But how do these symmetries transfer to the spacetime action?

群流形形式体系的大部分优势就体现于此: 如果考虑 $\widetilde{G}$ 上的“母”作用量 (20)，能保证的对称性是 $\widetilde{G}$ 的所有微分同胚，由沿 $\widetilde{G}$ 切向量 $\varepsilon = \varepsilon^A t_A$ 的李导数 ℓ_ε 生成。但这些对称性如何传递到时空作用量呢?

The variation of the group manifold action under diffeomorphisms generated by ℓ_ε is [1]

ℓ_ε is [1] 生成的微分同胚下群流形作用量的变分

$$\delta S = \int_{\widetilde{G}} \ell_\varepsilon (L \wedge \eta) = \int_{\widetilde{G}} (\ell_\varepsilon L) \wedge \eta + L \wedge \ell_\varepsilon \eta = 0 \tag{24}$$

modulo boundary terms. One has to vary the fields [2] in L as well as the submanifold embedded in $\widetilde{G}$: the sum of these two variations gives zero [3] on the group manifold action S . But what we need in order to have a spacetime interpretation of all the symmetries of S is really

模边界项。不仅需要变化嵌入在 $\widetilde{G}$ 中的子流形，还需要变分 L 中的场 [2] : 这两项变分之和对群流形作用量 S 给出零 [3] 。但要得到 S 所有对称性的时空诠释，我们实际需要的是

$$\delta S = \int_{\widetilde{G}} (\ell_\varepsilon L) \wedge \eta = 0 \tag{25}$$

[1] Recall $\ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ so that ℓ_ε (top form) = $d\,(\iota_\varepsilon$ top form))

[1] 回顾 $\ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ ， 因此 ℓ_ε (最高次形式) = $d\,(\iota_\varepsilon$ top form))

[2] Since ℓ_ε satisfies the Leibniz rule, $\ell_\varepsilon L$ can be computed by varying in turn all fields inside L .

[2] 由于 ℓ_ε 满足莱布尼茨律，可以通过依次变分 L 内部的所有场计算得到 $\ell_\varepsilon L$ 。

[3] In the following, the vanishing of action variations will always be understood modulo boundary terms.

[3] 下文默认，所有作用量变分的消失都是模边界项的。

If this holds, varying the fields ϕ inside L with the Lie derivative ℓ_ε as in (3), and then projecting on spacetime, yields spacetime variations

若该式成立，对 L 内部的场 ϕ 用李导数 ℓ_ε 做变分 (如式 (3) 所示)，再投影到时空，得到的时空变分

$$\delta\phi(x) = \ell_\varepsilon \phi(x, y, \theta)|_x \tag{26}$$

that leave the spacetime action (23) invariant. We have denoted by $|_x$ the projection on spacetime due to the integration on y and θ coordinates in (25). We call these variations spacetime invariances, since they leave invariant the spacetime action. They originate from the diffeomorphism invariance of the group manifold action and give rise to symmetries of the spacetime action (23) only when (25) holds. This happens if one of the following conditions is satisfied:

会保持时空作用量 (23) 不变。我们用 $|_x$ 表示由式 (25) 中对 y 和 θ 坐标积分得到的时空投影。我们称这些变分为时空不变性，因为它们保持时空作用量不变。它们起源于群流形作用量的微分同胚不变性，仅当式 (25) 成立时才会给出时空作用量 (23) 的对称性。满足下列任一条件即可保证式 (25) 成立:

- The Lie derivative on η vanishes:

- η 上的李导数为零:

$$\ell_\varepsilon \eta = 0 \tag{27}$$

- The spacetime projection of the Lie derivative of L is exact:

- L 李导数的时空投影是恰当形式:

$$(\ell_\varepsilon L)|_x = d\alpha \tag{28}$$

In this case, the variation (25)

在这种情况下，变分 (25)

$$\delta S = \int_{\widetilde{G}} (\ell_\varepsilon L) \wedge \eta = \int_{M^4} (\ell_\varepsilon L)|_x \tag{29}$$

vanishes after integration by parts. The requirement (28) is equivalent to

分部积分后消失。条件 (28) 等价于

$$(\iota_\varepsilon dL)|_x = d\alpha' \tag{30}$$

since $l_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ and $d\eta = 0$.

因为 $l_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ 且 $d\eta = 0$ 。

The Lagrangian L depends on the $\widetilde{G}$ -vielbein μ^A and its curvature R^A , so that also dL , after use of Bianchi identities, is expressed in terms of μ^A and R^A . Then condition (30) translates into a condition on the contractions $\iota_\varepsilon R^A$, i.e., a condition on the curvature components.

拉格朗日量 L 依赖于 $\widetilde{G}$ 标架 μ^A 及其曲率 R^A ，因此利用比安基恒恒等式后，dL 也可以用 μ^A 和 R^A 表示。于是条件 (30) 转化为缩并 $\iota_\varepsilon R^A$ 上的条件，也就是曲率分量上的条件。

Let us see how this works for superPoincaré supergravity.

让我们看看这对超庞加莱超引力是如何生效的。

Symmetries of $d = 4$ Supergravity

$d = 4$ 超引力的对称性

The symmetries of the spacetime action (spacetime invariances) are those generated by a Lie derivative ℓ_ε such that $\iota_\varepsilon dL|_x = d\alpha'$, cf. (30). We need to compute dL .

时空作用量的对称性 (时空不变性) 由李导数 ℓ_ε 生成，满足 $\iota_\varepsilon dL|_x = d\alpha'$ ，参见式 (30)。我们需要计算 dL 。

Using the Bianchi identities (14) and (15) and the definition of the torsion R^a in (10), we find:

利用比安基恒等式 (14)、(15)，以及式 (10) 中挠率 R^a 的定义，我们得到:

$$dL = 2R^{ab}R^cV^d\varepsilon_{abcd} + iR^{ab}\overline{\psi}\gamma^c\psi V^d\varepsilon_{abcd} + 4\overline{\rho}\gamma_5\gamma_a\rho V^a +$$

$$+\overline{\psi}\gamma_5\gamma_c\gamma_{ab}\psi R^{ab}V^c - 4\overline{\psi}\gamma_5\gamma_a\rho R^a - 2i\overline{\psi}\gamma_5\gamma_a\rho\overline{\psi}\gamma^a\psi \tag{31}$$

The gamma matrix identity

伽马矩阵恒等式

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + i\varepsilon_{abcd}\gamma_5\gamma^d \tag{32}$$

implies $\overline{\psi}\gamma_5\gamma_c\gamma_{ab}\psi = i\varepsilon_{abcd}\overline{\psi}\gamma^d\psi$, so that the second and the fourth term cancel in (31). Moreover, from the Fierz identity in Appendix ”Gamma Matrices in $d = 3 + 1$,” one deduces

可推导出 $\overline{\psi}\gamma_5\gamma_c\gamma_{ab}\psi = i\varepsilon_{abcd}\overline{\psi}\gamma^d\psi$, 因此第二项和第四项在式 (31) 中抵消。此外, 根据附录“$d = 3+1$ 中的伽马矩阵”中的费尔兹恒等式，可得

$$\gamma_a\psi\overline{\psi}\gamma^a\psi = 0 \tag{33}$$

and since $\overline{\psi}\gamma_5\gamma_a\rho = \overline{\rho}\gamma_5\gamma_a\psi$ also the last term in (31) vanishes due to (33). Therefore

又由于 $\overline{\psi}\gamma_5\gamma_a\rho = \overline{\rho}\gamma_5\gamma_a\psi$ ，结合式 (33)，式 (31) 中的最后一项也为零。因此

$$dL = 2R^{ab}R^cV^d\varepsilon_{abcd} + 4\overline{\rho}\gamma_5\gamma_a\rho V^a - 4\overline{\psi}\gamma_5\gamma_a\rho R^a \tag{34}$$

Lorentz Gauge Transformations

洛伦兹规范变换

It is immediate to see that if all curvatures are horizontal in the Lorentz directions (no ”legs” along ω), then indeed $\iota_{\varepsilon^{ab}t_{ab}} dL = 0$, and Lorentz transformations are a spacetime invariance of the supergravity action. This is essentially due to the absence of bare ω^{ab} in L . The general diffeomorphism formula (3) yields then the usual Lorentz transformations

不难看出，如果所有曲率在洛伦兹方向都是水平的 (沿 ω 没有“分量”)，那么确实有 $\iota_{\varepsilon^{ab}t_{ab}} dL = 0$ ，洛伦兹变换是超引力作用量的时空不变性。这本质上是因为 L 中不存在裸 ω^{ab} 。从一般微分同胚公式 (3) 便可得到常见的洛伦兹变换

$$\ell_{\varepsilon^{cd}t_{cd}} V^a = \varepsilon^a{}_b V^b \tag{35}$$

$$\ell_{\varepsilon^{cd}t_{cd}} \omega^{ab} = d\varepsilon^{ab} - \omega^a_c \varepsilon^{cb} + \omega^b_c \varepsilon^{ca} = D\varepsilon^{ab} \tag{36}$$

$$\ell_{\varepsilon^{cd}t_{cd}} \psi = \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\psi \tag{37}$$

We can check directly the invariance of the action under these variations: all curvatures and vierbeins appearing in (23) transform homogeneously, and Lorentz indices are contracted with Lorentz invariant tensors.

我们可以直接验证作用量在这些变分下的不变性:(23) 中出现的所有曲率和 vierbein 都是齐次变换的，且洛伦兹指标由洛伦兹不变张量缩并。

Spacetime Diffeomorphisms

时空微分同胚

Ordinary differentials along tangent vectors ∂_μ dual to dx^μ are invariances of the spacetime action, since (23) is an integral on a four-dimensional manifold of a four-form.

对偶于 dx^μ 的切向量 ∂_μ 的普通微分是时空作用量的不变性，因为 (23) 是四维流形上一个四形式的积分。

Supersymmetry Transformations

超对称变换

Differentials along tangent vectors t_α dual to ψ^α are spacetime invariances provided $\iota_\varepsilon dL|_x$ = total derivative with $\varepsilon = \varepsilon^\alpha t_\alpha$, that is to say,

对偶于 ψ^α 的切向量 t_α 的微分是时空不变量，前提是 $\iota_\varepsilon dL|_x =$ 带 $\varepsilon = \varepsilon^\alpha t_\alpha$ 的全导数，也就是说:

$$\iota_\varepsilon dL = 2\left(\iota_\varepsilon R^{ab}\right) R^c V^d \varepsilon_{abcd} + 2R^{ab}\left(\iota_\varepsilon R^c\right) V^d \varepsilon_{abcd} + 8\bar{\rho}\gamma_5\gamma_a\left(\iota_\varepsilon \rho\right) V^a$$

$$-4\bar{\varepsilon}\gamma_5\gamma_a\rho R^a - 4\bar{\psi}\gamma_5\gamma_a\left(\iota_\varepsilon\rho\right) R^a - 4\bar{\psi}\gamma_5\gamma_a\rho\left(\iota_\varepsilon R^a\right) = \text{ tot. der.} \tag{38}$$

once projected on spacetime. This is a condition for the contractions on the curvatures, and it is satisfied by

投影到时空上之后成立。这是曲率缩并需要满足的条件，该条件可由下式满足:

$$\iota_\varepsilon R^a = 0 \tag{39}$$

$$\iota_\varepsilon R^{ab} = -\varepsilon^{abef}\bar{\rho}_{ef}\gamma_5\gamma_g\varepsilon V^g - \varepsilon^{efg[a}\bar{\rho}_{ef}\gamma_5\gamma_g\varepsilon V^{b]} \equiv \bar{\theta}^{ab}_c\ \varepsilon V^c \tag{40}$$

$$\iota_\varepsilon \rho = 0 \tag{41}$$

Thus, we have supersymmetry invariance of the spacetime action if the curvatures have the following parametrization on a basis of two-forms:

因此，若曲率在二形式基上满足如下参数化，我们就得到时空作用量的超对称不变性:

$$R^a = R^a_{bc} V^b V^c \tag{42}$$

$$R^{ab} = R^{ab}_{cd} V^c V^d + \bar{\theta}^{ab}_c\ \psi V^c \tag{43}$$

$$\rho = \rho_{ab} V^a V^b \tag{44}$$

where we have taken into account also horizontality in the Lorentz directions. The conditions (39)-(41) are called ”rheonomic conditions,” and similarly (42)-(44) are called ”rheonomic parametrizations” of the curvatures.

其中我们还考虑了洛伦兹方向上的水平性。条件 (39)-(41) 被称为「流形条件」，类似地，(42)-(44) 被称为曲率的「流形参数化」。

The diffeomorphisms along $\varepsilon = \varepsilon^\alpha t_\alpha$ (supersymmetry transformations) act on the fields according to the general formula (3), where the contractions on the curvatures are given in (39)-(41):

沿 $\varepsilon = \varepsilon^\alpha t_\alpha$ 的微分同胚 (即超对称变换) 根据通式 (3) 作用在场量上，曲率缩并由 (39)-(41) 给出:

$$\ell_\varepsilon V^a = i\bar{\varepsilon}\gamma^a\psi \tag{45}$$

$$\ell_\varepsilon \omega^{ab} = \overline{\theta}_c^{ab} \ \varepsilon V^c \tag{46}$$

$$\ell_\varepsilon \psi = D\varepsilon \equiv d\varepsilon - \frac{1}{4}\omega^{ab}\gamma_{ab}\varepsilon \tag{47}$$

with $\overline{\theta}_c^{ab}$ defined in (40).

其中 $\overline{\theta}_c^{ab}$ 由式 (40) 定义。

Variational Principle and Field Equations

变分原理与场方程

The group manifold action (20) is a functional of L and of the embedded submanifold M , and therefore varying the action means varying both L and M . Varying M corresponds to varying η_M . Then the variational principle reads

群流形作用量 (20) 是 L 和嵌入子流形 M 的泛函，因此对作用量变分意味着同时对 L 和 M 变分。对 M 变分对应对 η_M 变分，变分原理可表述为

$$\delta S[L,M] = \int_{\widetilde{G}} (\delta L \wedge \eta_M + L \wedge \delta\eta_M) = 0. \tag{48}$$

Any (continuous) variation of M can be obtained by acting on η_M with a diffeomorphism generated by a Lie derivative ℓ_ξ . An arbitrary variation is generated by an arbitrary ξ vector, and the variational principle becomes

M 的任意 (连续) 变分都可通过李导数 ℓ_ξ 生成的微分同胚作用于 η_M 得到。任意变分由任意 ξ 向量生成，变分原理变为

$$\delta S[L,M] = \int_{\widetilde{G}} \left(\delta L \wedge \eta_M + L \wedge \ell_\xi \eta_M\right) = 0. \tag{49}$$

Since field variations in L and variation of M are independent, the two terms in (49) must vanish separately. From the vanishing of the first, we deduce

由于 L 的场变分与 M 的变分相互独立，(49) 中的两项必须分别为零。由第一项为零，我们推导出

$$\int_{\widetilde{G}} \left(\delta\phi \wedge \frac{\partial L}{\partial \phi} + d\delta\phi \wedge \frac{\partial L}{\partial (d\phi)}\right) \wedge \eta_M = 0 \tag{50}$$

where $L = L(\phi, d\phi)$ is considered a function of the one-form fields ϕ and their ”velocities” $d\phi$. A summation on all fields is understood. Integrating by parts and recalling $d\eta_M = 0$ yields

其中 $L = L(\phi, d\phi)$ 被视为一元场 ϕ 及其“速度”$d\phi$ 的函数。默认对所有场求和。分部积分并结合 $d\eta_M = 0$ 可得

$$\int_{\widetilde{G}} \delta\phi \wedge \left(\frac{\partial L}{\partial \phi} + d\frac{\partial L}{\partial (d\phi)}\right) \wedge \eta_M = 0 \tag{51}$$

and since the $\delta\phi$ are arbitrary, we find

且由于 $\delta\phi$ 是任意的，我们得到

$$\left(\frac{\partial L}{\partial \phi} + d\frac{\partial L}{\partial (d\phi)}\right) \wedge \eta_M = 0 \tag{52}$$

This must hold for any η_M (i.e., for generic embedding functions): we arrive therefore at equations that hold on the whole $\widetilde{G}$ and are the form version of the Euler-Lagrange equations:

这对任意 η_M 都成立 (即对任意一般嵌入函数): 因此我们得到在整个 $\widetilde{G}$ 上成立的方程，即欧拉-拉格朗日方程的形式版本:

$$\frac{\partial L}{\partial \phi} + d\frac{\partial L}{\partial (d\phi)} = 0 \tag{53}$$

If L is a d -form, these equations are $(d-1)$ -forms. Their content can be examined by expanding them along a complete basis of $(d-1)$ -forms in $\widetilde{G}$.

若 L 是 d 形式，则这些方程是 $(d-1)$ 形式。可以沿 $\widetilde{G}$ 中 $(d-1)$ 形式的一组完备基展开来分析它们的内容。

Requiring the vanishing of the second term in the variation (49) does not imply further equations besides the Euler-Lagrange field equations (53): indeed, this term vanishes on the shell of solutions of Euler-Lagrange equations. To prove it, notice that

要求变分 (49) 中的第二项为零，并不会得到欧拉-拉格朗日场方程 (53) 之外的额外方程: 事实上，该项在欧拉-拉格朗日方程的解壳上为零。证明如下，注意到

$$\int_{\widetilde{G}} L \wedge \ell_\xi \eta_M = -\int_{\widetilde{G}} \ell_\xi L \wedge \eta_M = 0 \text{ (on shell)} \tag{54}$$

because $\ell_\xi L$ is just a particular variation of L , under which the action remains stationary on shell.

因为 $\ell_\xi L$ 只是 L 的一种特殊变分，在解壳上作用量仍保持平稳。

Thus, the group manifold variational principle leads to the field equations (53), holding as $(d-1)$ -form equations on the whole $\widetilde{G}$:

因此，群流形变分原理导出了场方程 (53)，它作为 $(d-1)$ 形式方程在整个 $\widetilde{G}$ 上成立:

Note 1: The variational principle does not determine the embedding of M into $\widetilde{G}$.

注记 1: 变分原理无法确定 M 到 $\widetilde{G}$ 的嵌入。

Note 2: The field equations (53) are form equations and therefore invariant under the action of a Lie derivative. More precisely, if ϕ is a solution of (53), so is $\phi + \ell_\varepsilon \phi$: Lie derivatives generate symmetries of the field equations.

注记 2: 场方程 (53) 是形式方程, 因此在李导数作用下不变。更准确地说, 若 ϕ 是 (53) 的解, 则 $\phi + \ell_\varepsilon \phi$ 也是解: 李导数是场方程的对称性生成元。

Finally, we have the following:

最后，我们得到如下结论:

Theorem ($dL = 0$ (on shell)) . i.e., the Lagrangian, as a d-form on $\widetilde{G}$, is closed on shell. To prove it, recall that η_M is closed, so that on shell we find, cf. (54)

定理 ($dL = 0$ (on shell))。即拉格朗日量作为 $\widetilde{G}$ 上的 d 次形式，在解壳上是闭的。证明如下，回忆 η_M 是闭的，因此在解壳上我们得到，参见 (54)

$$0 = \int_{\widetilde{G}} L \wedge \ell_\xi \eta_M = \int_{\widetilde{G}} L \wedge d\iota_\xi \eta_M = -(-)^d \int_{\widetilde{G}} dL \wedge \iota_\xi \eta_M \tag{55}$$

ξ being arbitrary, this implies $dL = 0$ (on shell) [4]

由于 ξ 是任意的，这就推出 (在解壳上) $dL = 0$ [4]

Let us apply the preceding discussion to the superPoincaré supergravity example.

让我们将前述讨论应用到超庞加莱超引力例子中。

Supergravity Field Equations

超引力场方程

The variational equations (53) for the group manifold action (21) read

群流形作用量 (21) 的变分方程如下

$$2R^c V^d \varepsilon_{abcd} = 0 \tag{56}$$

$$2R^{ab} V^c \varepsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_d\rho = 0 \tag{57}$$

$$8\gamma_5\gamma_a\rho V^a - 4\gamma_5\gamma_a\psi R^a = 0 \tag{58}$$

obtained varying ω^{ab}, V^d , and ψ , respectively. The analysis proceeds as follows: we first expand the curvatures on a basis of two-forms [5] , with a set of coefficient functions that are a priori unknown:

分别通过对 ω^{ab}, V^d 和 ψ 变分得到。分析过程如下: 我们首先将曲率在二形式基 [5] 上展开，展开系数函数事先未知:

$$R^a = R^a_{bc} V^b V^c + \overline{\theta}^a_c \psi V^c + \overline{\psi} K^a \psi \tag{59}$$

$$R^{ab} = R^{ab}_{cd} V^c V^d + \overline{\theta}^{ab}{}_c \psi V^c + \overline{\psi} K^{ab} \psi \tag{60}$$

$$\rho = \rho_{ab} V^a V^b + H_c \psi V^c + \Omega_{\alpha\beta} \psi^\alpha \psi^\beta \tag{61}$$

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[4] In fact, this is just Stokes theorem applied to a $d+1$ -dimensional region of $\widetilde{G}$ bounded by two different hypersurfaces M and M' , with $\eta_{M'} = \eta_M + \ell_\xi \eta_M$.

[4] 事实上，这就是斯托克斯定理应用于 $\widetilde{G}$ 中由两个不同超曲面 M 和 M' 围成的 $d+1$ 维区域，其中满足 $\eta_{M'} = \eta_M + \ell_\xi \eta_M$ 。

[5] Assuming horizontality in the Lorentz directions. This amounts to consider configurations satisfying the Lorentz horizontality constraints on the curvatures.

[5] 假设洛伦兹方向上的水平性。这相当于考虑满足曲率上洛伦兹水平性约束的构型。

---

and then insert them into the field equations (56)-(58). These, being three-form equations, can be expanded on the basis $\psi\psi\psi, \psi\psi V, \psi VV$, and VVV . Their content is given below (the three lines correspond to the three equations of motion):

再将其代入场方程 (56)-(58)。这些方程是三形式方程，可以在基 $\psi\psi\psi, \psi\psi V, \psi VV$ 和 VVV 上展开。其内容如下所示 (三行对应三个运动方程):

$\psi\psi\psi$ sector:

$\psi\psi\psi$ 分支:

$$\Omega_{\alpha\beta} = 0 \tag{62}$$

$$0 = 0 \tag{63}$$

$$K^a = 0 \tag{64}$$

$\psi\psi V$ sector:

$\psi\psi V$ 分支:

$$2\overline{\psi}K^{ab}\psi V^c\varepsilon_{abcd} + 4\overline{\psi}\gamma_5\gamma_d H_c\psi V^c = 0 \tag{65}$$

$$0 = 0 \tag{66}$$

$$\overline{\theta}^{\,a}_{c} = 0 \tag{67}$$

ψVV sector:

ψVV 分支:

$$2\overline{\theta}^{\,ab}_{e}\,\psi V^e V^c\varepsilon_{abcd} + 4\overline{\psi}\gamma_5\gamma_d\rho_{ab}V^aV^b = 0 \tag{68}$$

$$0 = 0 \tag{69}$$

$$\gamma_5\gamma_a H_b\psi V^bV^a - 4\gamma_5\gamma_c\psi R^c{}_{ab}V^aV^b = 0 \tag{70}$$

VVV sector:

VVV 分支:

$$R^a_{bc} = 0 \tag{71}$$

$$R^{ac}{}_{bc} - \frac{1}{2}\delta^a_b R^{cd}{}_{cd} = 0 \tag{72}$$

$$\gamma^a\rho_{ab} = 0 \tag{73}$$

Inserting $R^a_{bc} = 0$ into (70) yields $H_c = 0$, which used in (65) gives $K^{ab} = 0$. Thus, the only nontrivial relation in the "outer" projections is (68), which determines $\theta^{ab}{}_c$ to be

将 $R^a_{bc} = 0$ 代入 (70) 得到 $H_c = 0$， 再将其用于 (65) 得到 $K^{ab} = 0$。因此，「外」投影中唯一非平凡的关系是 (68)， 它确定 $\theta^{ab}{}_c$ 为

$$\theta^{ab}_c = -\varepsilon^{abef}\overline{\rho}_{ef}\gamma_5\gamma_c - \delta^{[a}_c\varepsilon^{b]efg}\overline{\rho}_{ef}\gamma_5\gamma_g \tag{74}$$

in agreement with the $\theta^{abb}_{\ c}$ obtained from the condition (40). In this way, we arrive at the same curvature parametrizations (42)-(44) obtained in section "Symmetries of $d = 4$ Supergravity" by requiring spacetime supersymmetry invariance.

这与从条件 (40) 得到的 $\theta^{abb}_{\ c}$ 一致。通过这种方式，我们得到了与“$d = 4$ 超引力的对称性”一节中要求时空超对称性不变时相同的曲率参数化 (42)-(44)。

Finally, the VVV sector reproduces the (super)torsion equation and the propagation equations for the vierbein and the gravitino.

最终，VVV 分支重现了 (超) 挠率方程，以及 vierbein 和引力微子的传播方程。

Note: from the torsion equation

注: 从挠率方程

$$2R^a_{\mu\nu} \equiv \partial_\mu V^a_\upsilon - \partial_\nu V^a_\mu - \omega^a_{b,\mu} V^b_\upsilon + \omega^a_{b,\upsilon} V^b_\mu - i\overline{\psi}_\mu \gamma^a \psi_\upsilon = 0 \tag{75}$$

we can express the spin connection in terms of V and ψ , recovering second-order formalism:

我们可以用 V 和 ψ 表示自旋联络，得到二阶形式:

$$\omega_{ab,\mu} = \frac{1}{2} V_a^\upsilon V_b^\rho \eta_{cd} \left(\partial_{[\mu} V^c_{\upsilon]} V^d_\rho - \partial_{[\mu} V^c_{\rho]} V^d_\upsilon + \partial_{[\upsilon} V^c_{\rho]} V^d_\mu \right) +$$

$$+ \frac{i}{4} V_a^\upsilon V_b^\rho \left(\overline{\psi}_\mu \gamma_\upsilon \psi_\rho + \overline{\psi}_\upsilon \gamma_\rho \psi_\mu - \overline{\psi}_\rho \gamma_\mu \psi_\upsilon - (\upsilon \leftrightarrow \rho) \right) \tag{76}$$

Building Rules

构造规则

The group-geometric approach provides a systematic set of building rules [8] for constructing Lagrangians of supersymmetric theories:

群几何方法为构造超对称理论的拉格朗日量提供了一套系统的构造规则 [8]:

(1) Choose a Lie (super)algebra G , containing generators P_a that can be associated with d spacetime directions, and a Lorentz-like subalgebra H . Examples are the superPoincaré algebras in d dimensions or their uncontracted versions (orthosymplectic superalgebras $OSp\left(N \mid 2^{[d/2]}\right)$). The fields of the theory are the vielbein components of the soft group manifold $\widetilde{G}$.

(1) 选取一个李 (超) 代数 G ，它包含可对应 d 个时空方向的生成元 P_a ，以及一个类洛伦兹子代数 H 。例子包括 d 维超庞加莱代数，或其未缩并版本 (正交辛超代数 $OSp\left(N \mid 2^{[d/2]}\right)$)。该理论的场是软群流形 $\widetilde{G}$ 的标架分量。

(2) Construct the most general d -form on $\widetilde{G}$, by multiplying (with exterior products) one-form vielbein components σ^A and two-form curvatures R^A , without bare Lorentz connection and contracting indices with H -invariant tensors, so that the resulting Lagrangian is a Lorentz scalar.

(2) 通过外积乘上一元标架分量 σ^A 和二元曲率 R^A ，在 $\widetilde{G}$ 上构造最一般的 d -形式，不保留裸洛伦兹联络，并用 H 不变张量缩并指标，使最终得到的拉格朗日量是洛伦兹标量。

(3) Require that the variational equations admit the ”vacuum solution” $R^A = 0$, described by the vielbein of the rigid group manifold G .

(3) 要求变分方程存在由刚性群流形 G 的标架描述的“真空解”$R^A = 0$。

(4) The construction is greatly helped by scaling properties of the fields, dictated by the structure of the Lie (super)algebra G or equivalently by the Maurer-Cartan equations for the G vielbein. Consider, for example, the superPoincaré algebra: it is invariant under the rescalings $P_a \to \lambda P_a, M_{ab} \to M_{ab}, \bar{Q}_\alpha \to \lambda^{\frac{1}{2}} \bar{Q}_\alpha$. Then, the curvature definitions (10)-(12) are invariant under

(4) 李 (超) 代数 G 的结构，或等价地 G 标架的莫雷尔-嘉当方程，规定了场的标度性质，能极大简化构造过程。例如，超庞加莱代数在 $P_a \to \lambda P_a, M_{ab} \to M_{ab}, \bar{Q}_\alpha \to \lambda^{\frac{1}{2}} \bar{Q}_\alpha$ 重标度下不变，那么曲率定义 (10)-(12) 在下述变换下也不变

$$V^a \to \lambda V^a,\ \omega^{ab} \to \omega^{ab},\ \psi^\alpha \to \lambda^{\frac{1}{2}} \psi^\alpha \tag{77}$$

The field equations must be invariant under these rescalings, and therefore the action must scale homogeneously under (77). Since the Einstein-Hilbert term scales as λ^2 , all terms must scale in the same way, and this restricts the candidate terms in the Lagrangian.

场方程必须在这些重标度下不变，因此作用量必须在 (77) 下齐次标度。由于爱因斯坦-希尔伯特项的标度为 λ^2 ，所有项都必须有相同的标度，这就限制了拉格朗日量中的候选项。

(5) Finally, requiring that all terms have the same parity as the Einstein-Hilbert term further narrows the list of candidates.

(5) 最后，要求所有项都与爱因斯坦-希尔伯特项具有相同宇称，进一步缩小了候选项的范围。

The Lagrangian for $d = 4$ Supergravity

$d = 4$ 超引力的拉格朗日量

Following the above rules, one arrives at the $d = 4$ supergravity action (21). We recall here the steps of the procedure [8]. The most general Lagrangian four-form satisfying Rule 1 can at most contain two curvatures and is therefore of the type

遵循上述规则，我们可以得到 $d = 4$ 超引力作用量 (21)。我们在此回顾该流程的步骤 [8]。满足规则 1 的最一般四形式拉格朗日量最多包含两个曲率，因此形式为

$$L = R^A R^B \upsilon_{AB} + R^A \upsilon_A + \Lambda \tag{78}$$

with [6]

其中 [6]

$$R^A R^B \upsilon_{AB}^{(2)} = c_1 R^{ab} R^{cd} \varepsilon_{abcd} + c_2 R^{ab} R^{ab} + c_3 R^a R^a + c_4 \bar{\rho}\rho + c_5 \bar{\rho}\gamma_5\rho \tag{79}$$

The first two are total derivatives and are related to the Euler characteristic and to the Pontryagin number of M^4 . The last three can be reduced to linear terms in the curvatures plus total derivatives. Actually scaling invariance eliminates all the terms in (79) except $R^a R^a$, since the Einstein term scales as λ^2 . The torsion-squared term can be reduced to a linear term since

> 前两项是全导数，分别对应 M^4 的欧拉示性数和庞特里亚金数。后三项可约化为曲率的线性项加全导数。标度不变性会消去 (79) 中除 $R^a R^a$ 外的所有项，因为爱因斯坦项的标度为 λ^2 。挠率平方项可约化为线性项，原因是

$$R^a R^a = \left(DV^a - \frac{i}{2}\overline{\psi}\gamma^a\psi \right) R^a = d\left(V^a R^a\right) + V^a\left(-R^{ab}V^b + i\overline{\psi}\gamma^a\rho\right) - \frac{i}{2}\overline{\psi}\gamma^a\psi R^a$$

(80)

in virtue of the Bianchi identity (13). This leaves us with a Lagrangian of the form

> 这得益于比安基恒等式 (13)。由此我们得到如下形式的拉格朗日量:

$$L = \Lambda + v_{ab}R^{ab} + v_a R^a + \bar{v}\rho \tag{81}$$

where

> 其中

$$\Lambda = \alpha_1 \varepsilon_{abcd} V^a V^b V^c V^d + i\alpha_2 \varepsilon_{abcd}\overline{\psi}\gamma^{ab}\psi V^c V^d + i\alpha_3\overline{\psi}\gamma^{ab}\psi V^a V^b \tag{82}$$

$$v_{ab} = \beta_1 \varepsilon_{abcd} V^c V^d + \beta_2 V^a V^b + i\beta_3\overline{\psi}\gamma_{ab}\psi + i\beta_4\varepsilon_{abcd}\overline{\psi}\gamma^{cd}\psi \tag{83}$$

$$v_a = i\eta_1\overline{\psi}\gamma_a\psi \tag{84}$$

$$v = \delta_1\gamma_5\gamma_a\psi V^a + i\delta_2\gamma_a\psi V^a \tag{85}$$

are the most general Lorentz covariant terms. Notice that the only nonvanishing $\psi\psi$ currents are $\overline{\psi}\gamma^a\psi$ and $\overline{\psi}\gamma^{ab}\psi$. Correct λ^2 scaling of L drastically reduces the possible terms: $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = \beta_4 = 0$. Moreover parity implies $\beta_2 = \eta_1 = \delta_2 = 0$ (all terms must have the same parity as the Einstein term $R^{ab}V^c V^d \varepsilon_{abcd}$, i.e., must be pseudoscalars). Thus, we finally have

> 是最一般的洛伦兹协变项。注意，仅 $\overline{\psi}\gamma^a\psi$ 和 $\overline{\psi}\gamma^{ab}\psi$ 是非零的 $\psi\psi$ 流。对 L 正确的 λ^2 标度要求大幅减少了可能的项: $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = \beta_4 = 0$ 。此外，宇称要求给出 $\beta_2 = \eta_1 = \delta_2 = 0$ (所有项的宇称必须与爱因斯坦项 $R^{ab}V^c V^d \varepsilon_{abcd}$ 一致，即必须是赝标量)。因此，我们最终得到:

$$L = \beta_1 \varepsilon_{abcd} R^{ab} V^c V^d + \delta_1\overline{\psi}\gamma_5\gamma_a\rho V^a \tag{86}$$

---

[6] Repeated indices are contracted with the Minkowski flat metric.

[6] 重复指标用闵氏平度规缩并。

The requirement (3) that the vacuum be a solution of the field equations fixes the last parameter $a = \delta_1/\beta_1$. Indeed, the field equations obtained from the Lagrangian (86) by varying V^a, ω^{ab} , and ψ are, respectively,

真空是场方程解的要求 (3) 确定了最后一个参数 $a = \delta_1/\beta_1$ 。事实上，对拉格朗日量 (86) 分别变分 V^a, ω^{ab} 和 ψ 得到的场方程为:

$$2R^{ab}V^c\varepsilon_{abcd} + a\overline{\psi}\gamma_5\gamma_d\rho = 0 \tag{87}$$

$$2DV^cV^d\varepsilon_{abcd} + \frac{1}{4}a\overline{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \tag{88}$$

$$2a\gamma_5\gamma_a\rho V^a - a\gamma_5\gamma_a\psi R^a = 0 \tag{89}$$

To find the first is immediate; for the second, we only have to recall that varying ω^{ab} in R^{ab} yields $\delta R^{ab} = D\left(\delta\omega^{ab}\right)$ and that by integrating by parts the Lorentz covariant derivative D can be transferred on V^a . Finally, for the gravitino variation, we have

第一个结果可以直接得到；对第二个结果，我们只需记住: 对 R^{ab} 中的 ω^{ab} 做变分得到 $\delta R^{ab} = D\left(\delta\omega^{ab}\right)$，对分部积分后，洛伦兹协变导数 D 可以转移到 V^a 上。最后，对引力微子变分，我们有:

$$\frac{1}{a}\delta L = \left(\delta\overline{\psi}\right)\gamma_5\gamma_a D\psi V^a + \overline{\psi}\gamma_5\gamma_a D\left(\delta\psi\right)V^a = \tag{90}$$

$$= \left(\delta\overline{\psi}\right)\gamma_5\gamma_a D\psi V^a + \overline{\psi}\gamma_5\gamma_a\delta\psi DV^a + \delta\overline{\psi}\gamma_5\gamma_a D\psi V^a = \tag{91}$$

$$= 2\left(\delta\overline{\psi}\right)\gamma_5\gamma_a D\psi V^a - \delta\overline{\psi}\gamma_5\gamma_a\psi\left(R^a + \frac{i}{2}\overline{\psi}\gamma^a\psi\right) = \tag{92}$$

$$= \left(\delta\overline{\psi}\right)\left(2\gamma_5\gamma_a D\psi V^a - \gamma_5\gamma_a\psi R^a\right) \tag{93}$$

in virtue of $\overline{\psi}\gamma_5\gamma_a\left(\delta\psi\right) = -\left(\delta\overline{\psi}\right)\gamma_5\gamma_a\psi$ and the Fierz identity

这得益于 $\overline{\psi}\gamma_5\gamma_a\left(\delta\psi\right) = -\left(\delta\overline{\psi}\right)\gamma_5\gamma_a\psi$ 和菲尔兹恒等式

$$\gamma_a\psi\overline{\psi}\gamma^a\psi = 0 \tag{94}$$

Note that using the gamma-algebra identity

注意，利用伽马代数恒等式

$$\gamma_5\gamma_d\gamma_{ab} = 2\gamma_5\delta_{d[a}\gamma_{b]} - i\varepsilon_{abcd}\gamma^c \tag{95}$$

the variational equation (88) can be recast in the form:

变分方程 (88) 可以改写为如下形式:

$$2R^cV^d\varepsilon_{abcd} + \frac{i}{4}(4-a)\overline{\psi}\gamma_5\gamma_d\gamma_{ab}\psi V^d = 0 \tag{96}$$

so that the vacuum, defined by vanishing curvatures, is a solution of the field equations (5.21) only if $a = 4$.

因此，由零曲率定义的真空成为场方程 (5.21) 的解，当且仅当 $a = 4$ 。

In conclusion, applying the building rules with $G =$ superPoincaré yields the $N = 1, d = 4$ supergravity action (21).

综上，将构造规则应用于 $G =$ 超庞加莱群，就得到了 $N = 1, d = 4$ 超引力作用量 (21)。

Conclusions

结论

In this review, we have focused mostly on the logic of the group manifold approach, applied to a single example, i.e., $N = 1, d = 4$ supergravity. Comprehensive discussions on the applications of the method for the construction of supergravity theories in diverse dimensions can be found in the recent reviews [12,13].

在这篇综述中，我们主要关注将群流形方法的逻辑应用于单个实例，即 $N = 1, d = 4$ 超引力。关于该方法在不同维度超引力理论构建中的应用，可在近期综述文献 [12,13] 中找到全面讨论。

We list here some of the advantages/motivations:

我们在此列出该方法的部分优势/研究动机:

- All fields have a group-geometric origin, even if they are not all gauge fields.

  - 所有场都具有群几何起源，即便它们并非全都是规范场。

- All symmetries have a common origin as diffeomorphisms on $\widetilde{G}$.

  - 所有对称性都具有共同起源，即 $\widetilde{G}$ 上的微分同胚。

- There is a systematic procedure based on group geometry to construct actions, invariant under diffeomorphisms, and under gauge symmetries closing on a subgroup of G .

- 存在一套基于群几何的系统化流程，可以构建在微分同胚下不变，且在封闭于 G 某个子群的规范对称性下不变的作用量。

- Supersymmetry is formulated in a very natural way as a diffeomorphism in Grassmann directions of a supermanifold.

- 超对称被非常自然地表述为超流形格拉斯曼方向上的微分同胚。

- Closer contact is maintained with the usual component actions, whereas in the superfield formalism the actions look quite different. In fact, the group manifold action interpolates between the component and the superfield actions of the same supergravity theory (see [19-21]).

- 该方法与常规分量形式作用量联系更紧密,而超场形式的作用量看起来截然不同。实际上,群流形作用量可在同一超引力理论的分量作用量和超场作用量之间建立衔接 (参见文献 [19-21])。

- In the group manifold formulation of $d = 6$ supergravity [22] and $d = 10$ supergravity [23], the self-dual conditions for the three-form (in $d = 6$) and five-form (in $d = 10$) curvatures are a yield of the field equations in the respective superspaces and do not need to be imposed as external constraints.

- 在 $d = 6$ 超引力 [22] 和 $d = 10$ 超引力 [23] 的群流形表述中，三形式曲率 (对应 $d = 6$) 和五形式曲率 (对应 $d = 10$) 的自对偶条件是对应超空间场方程的自然结果，无需作为外生约束额外施加。

Finally, we recall some conceptual advances due to the group-geometric treatment of supergravity:

最后，我们回顾超引力群几何处理带来的若干概念层面进展:

- The generalization to p -form potentials, necessary to treat supergravity theories with p -form fields, in the framework of free differential algebras (FDA) [8,24- 27], and their dual formulation [28-31].

- 在自由微分代数 (FDA) 框架 [8,24-27] 中推广得到了 p 形式势，这是处理含 p 形式场的超引力理论所必需的，同时还给出了它们的对偶表述 [28-31]。

- The bridge between superspace and group manifold methods provided by superintegration, developed in Refs. [19-21].

- 文献 [19-21] 发展出的超积分方法，在超空间方法与群流形方法之间搭建了桥梁。

- A covariant Hamiltonian formalism, initially proposed in [32-34], based on the definition of field momenta as derivatives of the Lagrangian with respect to the exterior derivative of the fields, not involving a preferred direction (time). Recent developments [35,36] include the construction of all canonical symmetry generators for $N = 1, d = 4$ supergravity [36]. This covariant Hamiltonian formalism can also be generalized to a noncommutative (twisted) setting [37], describing noncommutative twisted (super)gravity [38,39].

- 协变哈密顿形式体系，最初由文献 [32-34] 提出，该体系将场动量定义为拉格朗日量对场外微分的偏导，不依赖偏好方向 (时间)。近期研究进展 [35,36] 包括构造了 $N=1, d=4$ 超引力的所有正则对称生成元 [36]。该协变哈密顿形式体系还可以推广到非对易 (扭) 框架 [37]，用以描述非对易扭 (超) 引力 [38,39]。

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Appendix: Group Manifold Geometry

附录: 群流形几何

This brief resumé is taken from section 2 of [11]. We start from a Lie algebra $\mathrm{Lie}(G)$, with generators T_A satisfying the commutation relations

这份简要概述摘自文献 [11] 的第 2 节。我们从李代数 $\mathrm{Lie}(G)$ 开始，其生成元 T_A 满足对易关系

$$[T_A, T_B] = C^C{}_{AB} T_C \tag{97}$$

For simplicity, we consider only usual Lie algebras. The extension to superalgebras is straightforward and only necessitates extra signs (e.g., anticommutators for fermionic generators) due to gradings.

为简化起见，我们这里仅讨论普通李代数。推广到超代数的过程是直接的，由于分次结构，仅需要额外引入符号 (例如费米子生成元采用反对易子)。

A generic group element $g \in G$ connected with the identity [7] can be expressed as

与单位元 [7] 连通的任意群元素 $g \in G$ 可以写为

$$g = \exp\left(y^A T_A\right) \equiv y \tag{98}$$

where y^A are the (exponential) coordinates of the group manifold. Each element of G is labeled by the coordinates y^A, and for notational economy, we denote it simply by y. Similarly, yx stands for $\exp\left(y^A T_A\right)\exp\left(x^B T_B\right)$, the product of two group elements, and by $(yx)^M$, we denote the corresponding coordinates.

其中 y^A 是群流形上的 (指数) 坐标。G 的每个元素都由坐标 y^A 标记，为简化记号，我们将其简记为 y。类似地，yx 代表两个群元素的乘积 $\exp\left(y^A T_A\right)\exp\left(x^B T_B\right)$，我们用 $(yx)^M$ 标记对应乘积的坐标。

Consider now $(yx)^M$ as a function [8] of x^A:

现在将 $(yx)^M$ 看作 x^A 的函数 [8] :

$$(yx)^M = y^M + e_A^M(y)\,x^A + e_{AB}^M(y)\,x^A x^B + \ldots \tag{99}$$

For infinitesimal x :

对于无穷小 x :

$$(yx)^M = y^M + \left(x^A t_A\right) y^M = \left(1 + x^A t_A\right) y^M,\ t_A \equiv e_A{}^N(y)\,\frac{\partial}{\partial y^N} \tag{100}$$

so that the t_A are a differential representation of the abstract generators T_A and satisfy therefore the same algebra:

因此 t_A 是抽象生成元 T_A 的微分表示，故而满足相同的代数:

$$[t_A, t_B] = C^C{}_{AB} t_C \tag{101}$$

The geometrical meaning of the components $e_A^N(y)$ in Eq. (99) is clear: consider the infinitesimal displacement $\delta_A y^M$ due to the (right) action of $1 + \varepsilon T_A(\varepsilon =$ infinitesimal parameter). Then

式 (99) 中分量 $e_A^N(y)$ 的几何意义很明确: 考虑 $1 + \varepsilon T_A(\varepsilon =$ (无穷小参数) 的 (右) 作用带来的无穷小位移 $\delta_A y^M$ ，则有

$$\delta_A y^M = \varepsilon e_A^M(y) \tag{102}$$

---

[7] Hereafter, G indicates the part of the group connected with the identity.

[7] 下文中，G 均指与单位元连通的群分支。

[8] Since G is a Lie group, this function is smooth.

[8] 由于 G 是李群，该函数是光滑的。

---

and the $\dim G$ vectors $e_A{}^M(y)$, A $=$ 1, … $\dim G$ are simply the tangent vectors at y in the direction of the displacements $\delta_A y^M$. It is customary to call tangent vector along the T_A direction the whole differential operator $t_A \equiv e_A{}^N(y)\frac{\partial}{\partial y^N}$. Note that $e_A{}^M$ is an invertible matrix, since the map $y \to yx$ is a diffeomorphism.

且 $\dim G$ 矢量 $e_A{}^M(y)$, A = 1, … $\dim G$ 正是 y 处沿位移 $\delta_A y^M$ 方向的切向量。通常将整个微分算符 $t_A \equiv e_A{}^N(y)\frac{\partial}{\partial y^N}$ 称为沿 T_A 方向的切向量。注意 $e_A{}^M$ 是可逆矩阵，因为映射 $y \to yx$ 是微分同胚。

The $t_A(y)$ span the tangent space of G at y : they form a contravariant basis. The "coordinate" basis, given by the vectors $\frac{\partial}{\partial y^N}$, is related to the t_A (the intrinsic basis) via the nondegenerate matrix ${e_A}^N$. The indices A, B,...are tangent space indices ("flat" indices) and are inert under y coordinate transformations. The indices M, N,... are coordinate indices ("world" indices) and do transform under coordinate transformations in the usual way (see later). Next, we define the one-forms $\sigma^A(y)$ as the duals of the t_A :

$t_A(y)$ 张成了 y 处 G 的切空间: 它们构成一个逆变基。由矢量 $\frac{\partial}{\partial y^N}$ 给出的"坐标"基，通过非退化矩阵 ${e_A}^N$ 与内蕴基 t_A 关联。指标 A, B⋯是切空间指标 ("平坦"指标)，在 y 坐标变换下不变。指标 M, N⋯是坐标指标 ("世界"指标)，按常规方式在坐标变换下变换 (见下文)。接下来我们定义形式 $\sigma^A(y)$ 为 t_A 的对偶:

$$\sigma^A(t_B) = \delta_A^B \tag{103}$$

The σ^A are a covariant basis (the intrinsic vielbein basis) for the dual of the tangent space, called cotangent space (the space of one-form). The "coordinate" cotangent basis dual to the $\frac{\partial}{\partial y^N}$ vectors is given by the differentials $dy^M\left(dy^M\left(\frac{\partial}{\partial y^N}\right) = \delta_N^M\right)$. The components of $\sigma^A(y)$ on the coordinate basis are denoted $e_M^A(y)$:

σ^A 是切空间的对偶空间 (余切空间，即一元形式空间) 的协变基 (内蕴标架基)。对偶于 $\frac{\partial}{\partial y^N}$ 矢量的"坐标"余切基由微分 $dy^M\left(dy^M\left(\frac{\partial}{\partial y^N}\right) = \delta_N^M\right.$ 给出。$\sigma^A(y)$ 在坐标基上的分量记为 $e_M^A(y)$:

$$\sigma^A(y) = e_M^A(y)\, dy^M \tag{104}$$

From the duality of the tangent and cotangent bases, we find

由切空间与余切基的对偶性，我们得到

$$e_M^A e_B^M = \delta_B^A \tag{105}$$

$$e_A^{-M} e_N^{-A} = \delta_N^M \tag{106}$$

Note 1: Substituting t_A by ${e_A}^N(y)\frac{\partial}{\partial y^N}$ into the commutator (101) leads to the differential condition on ${e_A}^M(y)$:

注 1: 将 t_A 替换为 ${e_A}^N(y)\frac{\partial}{\partial y^N}$ 代入对易式 (101)，得到 ${e_A}^M(y)$ 满足的微分条件:

$$-2e_{[A}{}^N e_{B]}{}^M \partial_N {e_M}^C = {C^C}_{AB} \tag{107}$$

Note 2: Computing the exterior derivative of σ^A , using equations (104) and (107) leads to the equations

注 2: 利用式 (104) 和 (107) 计算 σ^A 的外微分，得到方程

$$d\sigma^A + \frac{1}{2} C^A{}_{BC} \sigma^B \wedge \sigma^C = 0 \tag{108}$$

These are called Maurer-Cartan (MC) equations and provide a dual formulation of Lie algebras in terms of the one-form σ^A . It is immediate to verify that the closure of the exterior derivative $\left(d^2 = 0\right)$ is equivalent to the Jacobi identities for the structure constants:

这些被称为莫雷尔-嘉当 (MC) 方程，它给出了李代数用一元形式 σ^A 表示的对偶形式。可以直接验证，外微分 $\left(d^2 = 0\right)$ 的封闭性等价于结构常数满足雅可比恒等式:

$$C^A{}_{B[C} C^B{}_{DE]} = 0 \tag{109}$$

(apply d to Eq. (108)).

(对式 (108) 应用 d)。

Note 3: Defining $\sigma(y) \equiv \sigma^A(y) T_A$, the MC equations (108) take the form

注 3: 定义 $\sigma(y) \equiv \sigma^A(y) T_A$ 后，MC 方程 (108) 可写为

$$d\sigma + \sigma \wedge \sigma = 0 \tag{110}$$

The Lie-valued one-form $\sigma(y)$ can also be constructed directly from the group element y :

李值一元形式 $\sigma(y)$ 也可以直接从群元 y 构造得到:

$$\sigma(y) = y^{-1} dy \tag{111}$$

It is easy to verify that (111) satisfies the MC equation (110) (use $dy^{-1} = -y^{-1} dy y^{-1}$) . Moreover, it takes the same value as $e_M{}^A dy^M T_A$ at the origin $y = 0$. Indeed, from the definition of $e_A{}^M$ in Eq. (99), one sees that $e_A{}^M(y = 0) = \delta_A^M$, and therefore $e_M{}^A(0)\, dy^M T_A = dy^A T_A$. This value coincides with $y^{-1}dy|_{y=0}$ since $y^{-1}|_{y=0} =$ [group unit] and $dy|_{y=0} = dy^A T_A$ (from (98)). This observation suffices to conclude that $y^{-1}dy$ is equal to $e_M{}^A(y)\, dy^M T_A$.

容易验证 (111) 满足 MC 方程 (110)，验证过程可使用 $dy^{-1} = -y^{-1} dy y^{-1}$)。此外，在原点 $y = 0$ 处它与 $e_M{}^A dy^M T_A$ 取值相同。实际上，由式 (99) 中 $e_A{}^M$ 的定义可知 $e_A{}^M(y = 0) = \delta_A^M$ ，因此有 $e_M{}^A(0)\, dy^M T_A = dy^A T_A$ 。该值与 $y^{-1}dy|_{y=0}$ 一致，因为 $y^{-1}|_{y=0} =$ [group unit] 且 $dy|_{y=0} = dy^A T_A$ ，这来自式 (98)。该观察足以证明 $y^{-1}dy$ 等于 $e_M{}^A(y)\, dy^M T_A$ 。

Soft Group Manifold

软群流形

Consider a smooth deformation $\widetilde{G}$ of the group manifold G . Its vielbein field is given by the intrinsic cotangent basis, defined for any differentiable manifold:

考虑群流形 G 的一个光滑形变 $\widetilde{G}$ 。其标架场由内蕴余切基给出，该定义适用于任意微分流形:

$$\mu^A(y) = \mu^A_M(y)\, dy^M \tag{112}$$

where we use the symbol μ for the soft vielbein. In general, μ^A does not satisfy the MC equations anymore, so that

其中我们用符号 μ 表示软标架。一般而言，μ^A 不再满足毛雷尔-嘉当方程，因此有

$$d\mu^A + \frac{1}{2} C^A{}_{BC}\mu^B \wedge \mu^C \equiv R^A \neq 0 \tag{113}$$

The extent of the deformation $G \to \widetilde{G}$ is measured by the curvature two-form R^A . $R^A = 0$ implies $\mu^A = \sigma^A$ and vice versa.

形变 $G \to \widetilde{G}$ 的偏离程度由曲率二次形式 R^A 度量。$R^A = 0$ 蕴含 $\mu^A = \sigma^A$ ，反之亦然。

Applying the external derivative d to the definition (113), using $d^2 = 0$ and the Jacobi identities on $C^A{}_{BC}$, yields the Bianchi identities

对式 (113) 的定义应用外导数 d ，利用 $d^2 = 0$ 和作用于 $C^A{}_{BC}$ 的雅可比恒等式，可推导出比安基恒等式

$$(\nabla R)^A \equiv dR^A - C^A{}_{BC}R^B \wedge \mu^C = 0 \tag{114}$$

Diffeomorphisms and Lie Derivative

微分同胚与李导数

First, we discuss the variation under diffeomorphisms of the vielbein field $\mu^A(y)$:

首先，我们讨论标架场 $\mu^A(y)$ 在微分同胚下的变分:

$$\mu^A(y + \delta y) - \mu^A(y) = \delta\left[\mu^A_M(y)\, dy^M\right] =$$

$$= \left(\partial_N \mu^A_M\right)\delta y^N dy^M + \mu^A_M\left(\partial_N \delta y^M\right) dy^N =$$

$$= dy^N\left[\partial_N \delta y^A + \delta y^M\left(\partial_M \mu^A_N - \partial_N \mu^A_M\right)\right] =$$

$$= d\delta y^A - 2\mu^B \delta y^C (d\mu^A)_{BC} = d\left(\iota_{\delta y}\mu^A\right) + \iota_{\delta y} d\mu^A \tag{115}$$

where

其中

$$\delta y^A \equiv \delta y^M \mu_M^A, \ \delta y \equiv \delta y^M \partial_M, \ d\mu^A \equiv \left(d\mu^A\right)_{BC} \mu^B \wedge \mu^C, \tag{116}$$

and the contraction ι_t along a tangent vector t is defined on p -forms

沿切向量 t 的缩并 ι_t 定义在 p -形式上

$$\omega_{(p)} = \omega_{B_1 \ldots B_p} \mu^{B_1} \wedge \ldots \wedge \mu^{B_p} \tag{117}$$

as

$$\iota_t \omega_{(p)} = p t^A \omega_{A B_2 \ldots B_p} \mu^{B_2} \wedge \ldots \wedge \mu^{B_p} \tag{118}$$

Note that ι_t maps p -forms into $(p-1)$ -forms. The operator

注意 ι_t 将 p -形式映射为 $(p-1)$ -形式。该算子

$$\ell_t \equiv d\iota_t + \iota_t d \tag{119}$$

is called the Lie derivative along the tangent vector t and maps p -forms into p -forms. As shown in Eq. (115), the Lie derivative of the one-form μ^A along δy gives its variation under the diffeomorphism $y \to y + \delta y$. This holds true for any p -form.

称为沿切向量 t 的李导数，它将 p -形式映射为 p -形式。如式 (115) 所示，1-形式 μ^A 沿 δy 的李导数给出其在微分同胚 $y \to y + \delta y$ 下的变分，这对任意 p -形式都成立。

We now rewrite the variation $\delta\mu^A$ of Eq. (115) in a suggestive way, by adding and subtracting $C^A{}_{BC}\mu^B\delta y^C$:

我们现在将式 (115) 的变分 $\delta\mu^A$ 改写为更直观的形式，通过加减 $C^A{}_{BC}\mu^B\delta y^C$ 得到:

$$\delta\mu^A = d\delta y^A + C^A{}_{BC}\mu^B\delta y^C - 2\mu^B\delta y^C \left(d\mu^A\right)_{BC} - C^A{}_{BC}\mu^B\delta y^C \tag{120}$$

$$= (\nabla\delta y)^A + \iota_{\delta y} R^A \tag{121}$$

where we have used the definition (113) for the curvature, and the G -covariant derivative ∇ acts on δy^A as

其中我们用到了曲率的定义 (113)，且 G 协变导数 ∇ 对 δy^A 的作用为

$$(\nabla\delta y)^A \equiv d\mu^A + C^A{}_{BC}\mu^B\delta y^A \tag{122}$$

The Algebra of Lie Derivatives

李导数代数

The algebra of diffeomorphisms is given by the commutators of Lie derivatives:

微分同胚代数由李导数对易子给出:

$$\left[\ell_{\varepsilon_1^A t_A}, \ell_{\varepsilon_2^B t_B}\right] = \ell_{\varepsilon_3^C t_C} \tag{123}$$

with

其中

$$\varepsilon_3^C = \varepsilon_1^A \partial_A \varepsilon_2^C - \varepsilon_2^A \partial_A \varepsilon_1^C - 2\varepsilon_1^A \varepsilon_2^B \mathcal{R}_{AB}^C \tag{124}$$

and

且

$$\mathcal{R}_{AB}^C \equiv R_{AB}^C - \frac{1}{2} C_{AB}^C \tag{125}$$

The components R_{BC}^A are defined by $R^A = R_{BC}^A \mu^B \wedge \mu^C$. The closure of the algebra requires the Bianchi identities (114), which we can rewrite in the form

分量 R_{BC}^A 由 $R^A = R_{BC}^A \mu^B \wedge \mu^C$ 定义。代数的封闭性要求比安基恒等式 (114)，我们可以将其改写为如下形式

$$\partial_{[B} \mathcal{R}_{CD]}^A + 2\mathcal{R}_{E[B}^A \mathcal{R}_{CD]}^E = 0 \tag{126}$$

To prove (123), just apply both sides of the equation to the soft vielbein μ^A .

要证明 (123)，只需将方程两边同时作用于软标架 μ^A 即可。

Integration on Supermanifolds: Integral Forms

超流形上的积分: 积分形式

In this Appendix, taken from section 4 of [12], we recall basic results in superman-ifold integration (see, for example, [40] for a recent review or [41] for a textbook) and new developments concerning integral forms, discussed in Refs. [19-21].

本附录摘自文献 [12] 的第 4 节，我们回顾超流形积分的基本结果 (近期综述可参见 [40]，教材可参见 [41])，以及参考文献 [19-21] 中讨论的关于积分形式的新进展。

We have defined the supergravity action (21) as an integral of a top form on the superPoincaré group manifold. We have given explicitly only the four-form Lagrangian, postponing the precise expression of η_{M^4} to the present Appendix. In fact, in the supergravity case, we have tacitly assumed typical properties of bosonic integration, as, for example, the existence of a top form and Stokes' theorem. Here, we want to justify these assumptions and give a short account of superintegration theory.

我们已将超引力作用量 (21) 定义为超庞加莱群流形上最高次形式的积分。我们此前仅显式给出了四形式拉格朗日量，将 η_{M^4} 的精确表达式推迟到本附录给出。实际上，在超引力情形中，我们默认假定了玻色积分的典型性质，例如最高次形式的存在性和斯托克斯定理。本文在此旨在证明这些假设的合理性，并简要介绍超积分理论。

The construction of actions invariant under diffeomorphisms is solved ab initio in ordinary integration theory by form integration. The integral of a d -form

在普通积分理论中，微分同胚不变作用量的构造从一开始就通过形式积分解决了。d 形式的积分

$$\omega^{(d)} = \omega_{[\mu_1 \cdots \mu_d]}(x)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \tag{127}$$

on a d -dimensional manifold M^d is defined by

在 d 维流形 M^d 上由下式定义

$$I = \int_{M^d} \omega^{(d)} \equiv \int_{M^d} \frac{1}{d!} \omega_{[\mu_1 \cdots \mu_d]}(x)\, \varepsilon^{\mu_1 \cdots \mu_d} d^d x, \tag{128}$$

i.e., by usual (Riemann-Lebesgue) integration on M^d of the function $\frac{1}{d!}\omega_{[\mu_1 \cdots \mu_d]}(x)$ $\varepsilon^{\mu_1 \cdots \mu_d}$, where $\varepsilon^{\mu_1 \cdots \mu_d}$ is the Levi-Civita antisymmetric symbol in the coordinate basis, a tensor density of weight -1 . Therefore,

即，通过对函数 $\frac{1}{d!}\omega_{[\mu_1 \cdots \mu_d]}(x)$ $\varepsilon^{\mu_1 \cdots \mu_d}$ 在 M^d 上做常规 (黎曼-勒贝格) 积分，其中 $\varepsilon^{\mu_1 \cdots \mu_d}$ 是坐标基中的列维-奇维塔反对称符号，是权为-1 的张量密度。因此，

$$\varepsilon^{\mu_1 \cdots \mu_d} d^d x = \varepsilon^{\mu_1 \cdots \mu_d} dx^1 \wedge \cdots \wedge dx^d \tag{129}$$

is a tensor, and the integrand of (128) is a scalar.

是一个张量，(128) 的被积函数是标量。

As in the previous sections, we can consider infinitesimal diffeomorphisms as active transformations, generated by the Lie derivative $\ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$. Then the form integral (128) transforms as

和前面几节一样，我们可以将无穷小微分同胚视为由李导数 $\ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon$ 生成的主动变换。那么形式积分 (128) 的变换为

$$\delta I = \int_{M^d} \ell_\varepsilon \omega^{(d)} = \int_{M^d} (\iota_\varepsilon d + d\iota_\varepsilon)\, \omega^{(d)} = 0 \tag{130}$$

since $d\omega^{(d)} = 0$ ($\omega^{(d)}$ is a top form) and $\int_{M^d} d\,(\iota_\varepsilon \omega) = 0$ for appropriate boundary conditions. Thus, we have checked invariance of the form integral under infinitesimal differentials generated by the Lie derivative. Note that the existence of a top form, namely, the fact that a d -form is closed on M^d , is crucial to ensure action invariance under differentials.

(因为 $d\omega^{(d)} = 0$($\omega^{(d)}$ 是最高次形式)，在合适的边界条件下 $\int_{M^d} d\,(\iota_\varepsilon \omega) = 0$ 成立。由此我们验证了形式积分在李导数生成的无穷小微分同胚下的不变性。注意，最高次形式的存在，即 d 形式在 M^d 上是闭的这一事实，对保证微分同胚下作用量的不变性至关重要。

Can we generalize form integration to supermanifolds and use it to construct actions automatically invariant under superdiffeomorphisms? The answer to both questions is affirmative.

我们能否将形式积分推广到超流形，并用它构造自动满足超微分同胚不变性的作用量？两个问题的答案都是肯定的。

In analogy with the bosonic case, integration on forms living on supermanifolds is defined via integration of functions in superspace. Consider a function $\Phi(x,\theta)$, defined on a supermanifold $M^{d|m}$ with d bosonic coordinates x and m fermionic (anticommuting) coordinates θ^α . It is called a superfield and can be expanded in the θ^α coordinates:

与玻色情形类似，超流形上形式的积分通过超空间上函数的积分定义。考虑定义在具有 d 个玻色坐标 x 和 m 个费米 (反对易) 坐标 θ^α 的超流形 $M^{d|m}$ 上的函数 $\Phi(x,\theta)$ 。该函数称为超场，可按 θ^α 坐标展开:

$$\Phi(x,\theta) = \phi(x) + \phi_{\alpha_1}(x)\,\theta^{\alpha_1} + \phi_{\alpha_1\alpha_2}(x)\,\theta^{\alpha_1}\theta^{\alpha_2} + \cdots + \phi_{\alpha_1\cdots\alpha_m}(x)\,\theta^{\alpha_1}\cdots\theta^{\alpha_m}$$

(131)

The functions $\phi_{\alpha_1\cdots\alpha_p}(x)$ are called superfield components and have antisym-metrized indices due to the anticommuting θ ’ s in the expansion (131). The integral of the superfield on $M^{d|m}$ is defined by Berezin integration:

函数 $\phi_{\alpha_1\cdots\alpha_p}(x)$ 称为超场分量，由于展开式 (131) 中反对易的 θ ，分量带有反对称化指标。超场在 $M^{d|m}$ 上的积分由别列津积分定义为:

$$\int_{M^{d|m}} \Phi(x,\theta)\, d^d x d^m \theta \equiv \int_{M^d} \frac{1}{m!} \phi_{\alpha_1\cdots\alpha_m}(x)\, \varepsilon^{\alpha_1\cdots\alpha_m} d^d x \tag{132}$$

Only the highest component of Φ (corresponding to the maximal number of θ s) enters the integral on M^d .

只有 Φ 的最高分量 (对应最多数量的 θ) 会出现在 M^d 的积分中。

Note the striking similarity between the two integrals (128) and (132). In fact, we can define form integration in terms of Berezin integration. Consider the differentials dx in the d -form (127) as anticommuting coordinates $\xi^\mu = dx^\mu$, so that $\omega^{(d)}$ becomes a function of x and ξ :

请注意积分 (128) 和 (132) 之间惊人的相似性。事实上，我们可以基于贝雷津积分定义形式积分。将 dx -形式 (127) 中的微分 d 视作反交换坐标 $\xi^\mu = dx^\mu$ ，由此 $\omega^{(d)}$ 可变为 x 和 ξ 的函数:

$$\omega^{(d)}(x,\xi) = \omega_{[\mu_1 \cdots \mu_d]}(x)\, \xi^{\mu_1} \cdots \xi^{\mu_d} \tag{133}$$

Then its Berezin integral on $M^{d|d}$ exactly yields the form integral (128). This observation is the key for a definition of superform integration on supermanifolds.

那么它对 $M^{d|d}$ 的别列津积分恰好给出形式积分 (128)。这一观察是定义超流形上超形式积分的核心。

A natural generalization of a bosonic top form (127) is a $(d+m)$ -superform:

玻色顶部形式 (127) 的一个自然推广是 $(d+m)$ 超形式:

$$\omega^{(d+m)}(x,\theta) = \omega_{[\mu_1 \cdots \mu_d]\{\alpha_1 \cdots \alpha_m\}}(x,\theta)\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_m}$$

(134)

Note that the $d\theta$ differentials commute (since the θ s are anticommuting), so that the indices α_i are symmetrized. For this reason, $\omega^{(d+m)}$ cannot be a top form: a superform can have an arbitrary number of $d\theta$ differentials, and its exterior derivative does not vanish. Let's ignore for the moment this difficulty and try to define a superform integral. Inspired by the observation in the preceding paragraph, we consider the superform $\omega^{(d+m)}(x,\theta)$ as a function of $x, \theta, dx, d\theta$, i.e., a function of the commuting variables $x, d\theta$ and anticommuting variables θ, dx . Its integral can be defined by Berezin integration on θ, dx , and usual Riemann-Lebesgue integration on $x, d\theta$. Here, a second difficulty arises: the ordinary integration on the $u = d\theta$ coordinates produces integrals of the type

注意 $d\theta$ 个微分是交换的 (因为 θ 是反对易的)，因此指标 α_i 需要对称化。因此，$\omega^{(d+m)}$ 不可能是顶部形式: 超形式可以包含任意数量的 $d\theta$ 微分，且其外导数不为零。我们暂时搁置这个困难，尝试定义超形式积分。受前一段观察的启发，我们将超形式 $\omega^{(d+m)}(x,\theta)$ 看作 $x, \theta, dx, d\theta$ 的函数，即交换变量 $x, d\theta$ 和反对易变量 θ, dx 的函数。它的积分可定义为对 θ, dx 的别列津积分，对 $x, d\theta$ 的常规黎曼-勒贝格积分。这里出现了第二个困难: 对 $u = d\theta$ 坐标的常规积分会得到如下类型的积分

$$\int u^m d^m u \tag{135}$$

and there is no algorithmic way to assign a C -number to it. For the integral on the even variables $u = d\theta$ to make sense, the integrand must have a compact support as a function of u . For this reason, we consider functions of the $d\theta$ s which are distributions in $d\theta$ with support at the origin:

且不存在算法方式为它赋值一个 C 数。要让偶变量 $u = d\theta$ 上的积分有意义，被积函数作为 u 的函数必须具有紧支集。因此，我们考虑在原点带支集的 $d\theta$ 上分布，作为 $d\theta$ 的函数:

$$\omega(x, \theta, dx, d\theta) = \omega_{[\mu_1 \cdots \mu_d]}(x, \theta)\, dx^{\mu_1} \cdots dx^{\mu_d} \delta\left(d\theta^1\right) \cdots \delta\left(d\theta^m\right) \tag{136}$$

These "functions" can be integrated on the supermanifold $M^{d+m|d+m}$ spanned by the $d+m$ bosonic variables $x, d\theta$ and $d+m$ fermionic variables dx, θ . The integral

这些" 函数" 可以在由 $d+m$ 个玻色变量 $x, d\theta$ 和 $d+m$ 个费米子变量 dx, θ 张成的超流形 $M^{d+m|d+m}$ 上积分。该积分

$$\int_{M^{d+m|d+m}} \omega(x, \theta, dx, d\theta)\, d^d x d^m \theta d^d(dx)\, d^m(d\theta) \tag{137}$$

is defined by Berezin integration on the odd variables dx, θ and usual Riemann-Lebesgue integration on the even variables $x, d\theta$. Carrying out integration on the variables dx and $d\theta$, the integral becomes

定义为对奇变量 dx, θ 的别列津积分，对偶变量 $x, d\theta$ 的常规黎曼-勒贝格积分。完成对变量 dx 和 $d\theta$ 的积分后，积分变为

$$\int_{M^{d|m}} \omega_{[\mu_1 \cdots \mu_d]}(x, \theta)\, \varepsilon^{\mu_1 \cdots \mu_d} d^d x d^m \theta \tag{138}$$

This integral can also be seen as an integral of the form:

这个积分也可以写成如下形式:

$$\omega^{d|m} = \omega_{[\mu_1 \cdots \mu_d]}(x, \theta)\, \delta\left(u^1\right) \cdots \delta\left(u^m\right) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge du^1 \wedge \cdots \wedge du^m$$

(139)

where the even variables u are the differentials $d\theta$. Indeed, let us integrate this form with the recipe of considering it a function of x, θ, u and of the differentials dx, du , and then using Berezin and Riemann integration according to the odd or even grading of the variables. The result coincides with (138).

其中偶变量 u 就是微分 $d\theta$ 。事实上，我们按照将该形式看作 x, θ, u 和微分 dx, du 的函数，再根据变量的奇偶分次分别使用别列津积分和黎曼积分的方法对其积分，结果与 (138) 一致。

Thus, the form $\omega^{d|m}$ can be integrated, even if it contains $d\theta$ differentials. We achieve this by confining the $d\theta$ ' s inside delta functions and in this way overcome the first difficulty encountered with the superforms (134). But can $\omega^{d|m}$ overcome also the second difficulty and be a top form? The answer is yes: the dx and du differentials are all anticommuting, so that their number in $\omega^{d|m}$ is already maximal, and multiplying it by $d\theta$ differentials gives zero because of the presence of the deltas. Therefore, $d\omega^{d|m} = 0$, and $\omega^{d|m}$ is a bona fide top form. Since it can be integrated and it is a top form, $\omega^{d|m}$ is called an integral top form.

因此，即使 $\omega^{d|m}$ 包含 $d\theta$ 个微分，它仍然可以被积分。我们通过将 $d\theta$ 限制在德尔塔函数内部来实现这一点，以此解决超形式遇到的第一个难点 (134)。但 $\omega^{d|m}$ 能否也克服第二个难点，成为一个 top 形式呢？答案是肯定的: dx 和 du 微分都是反交换的，因此它们在 $\omega^{d|m}$ 中的数量已经达到最大，而由于德尔塔函数的存在，再乘以 $d\theta$ 个微分会得到零。因此，$d\omega^{d|m}=0$，$\omega^{d|m}$ 是一个真正的 top 形式。由于它可积分且是 top 形式，因此 $\omega^{d|m}$ 被称为积分 top 形式。

Finally, using the notation

最后，使用记号

$$\delta\left(u^1\right)\cdots\delta\left(u^m\right)du^1\wedge\cdots\wedge du^m\equiv\delta\left(u^1\right)\wedge\cdots\wedge\delta\left(u^m\right) \tag{140}$$

the integral top form can be rewritten (using $u=d\theta$):

积分 top 形式可以重写为 (利用 $u=d\theta$):

$$\omega^{d|m}=\omega_{[\mu_1\cdots\mu_d]}(x,\theta)\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_d}\wedge\delta\left(d\theta^1\right)\wedge\cdots\wedge\delta\left(d\theta^m\right) \tag{141}$$

or also

也可以写为

$$\omega^{d|m}=\omega_{[\mu_1\cdots\mu_d][\alpha_1\cdots\alpha_m]}(x,\theta)\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_d}\wedge\delta\left(d\theta^{\alpha_1}\right)\wedge\cdots\wedge\delta\left(d\theta^{\alpha_m}\right)$$

(142)

where indices α are antisymmetrized since the $\delta\left(d\theta^{\alpha}\right)$ anticommute and

其中由于 $\delta\left(d\theta^{\alpha}\right)$ 反交换，指标 α 做反对称化，且

$$m!\,\omega_{[\mu_1\cdots\mu_d]}\equiv\omega_{[\mu_1\cdots\mu_d][1\cdots m]} \tag{143}$$

In this notation, μ and α indices play a similar role and are both antisymmetrized. The numbers d and dm are, respectively, called the form number and the picture number, and for integral top forms, they coincide with the numbers of bosonic and fermionic dimensions of the supermanifold $M^{d|m}$.

在该记号下，μ 指标与 α 指标作用相似，都做反对称化。数 d 和 dm 分别被称为形式数和影数，对于积分 top 形式，它们分别等于超流形 $M^{d|m}$ 的玻色维度数与费米维度数。

We call ”superforms” the forms of the kind (134), with dx and $d\theta$ differentials, without $\delta\left(d\theta\right)$ s. Thus, superforms have a form number that counts the $dx, d\theta$ differentials and zero picture number. For example, the Lagrangian in (21) is a superform $L^{4|0}$.

我们将形如 (134)、含有 dx 和 $d\theta$ 微分且不含 $\delta\left(d\theta\right)$ 的形式称为“超形式”。因此，超形式的形式数计数 $dx, d\theta$ 微分，且影数为零。例如，(21) 中的拉格朗日量就是超形式 $L^{4|0}$ 。

Integration on Submanifolds of Supermanifolds

超流形子流形上的积分

Supergravity actions on supergroup manifolds $\widetilde{G}$ are given by integrals of a d -form Lagrangian L on a d -dimensional (bosonic) submanifold M^d of $\widetilde{G}$. They can be written as integrals on the whole $\widetilde{G}$ of the Lagrangian multiplied by an appropriate Poincaré dual η_{M^d} of M^d , such that $L \wedge \eta_{M^d}$ becomes an integral top form. Let us see how this works for $N = 1, d = 4$ supergravity.

超流形 $\widetilde{G}$ 上的超引力作用量由 d -形式拉氏量 L 在 $\widetilde{G}$ 的 d 维 (玻色) 子流形 M^d 上的积分给出。它们可以改写为整个 $\widetilde{G}$ 上的积分: 拉氏量乘上 M^d 的一个合适的庞加莱对偶 η_{M^d} ， 使得 $L \wedge \eta_{M^d}$ 成为可积的最高次形式。下面我们来看这在 $N = 1, d = 4$ 超引力中是如何运作的。

The supergravity Lagrangian in (21) is a $(4 \mid 0)$ superform. For simplicity, we now assume that fields satisfy the Lorentz horizontality constraints on all the curvatures and thus effectively depend only on the superspace coordinates x^μ, θ^α , with $\mu = 1, \ldots 4, \alpha = 1, \ldots 4$. Then $\widetilde{G}$ is $M^{4|4}$ superspace, and only integral top forms of type (4|4) can be integrated on $M^{4|4}$. We therefore need a Poincaré dual of type $(0 \mid 4)$, so that

(21) 中的超引力拉氏量是一个 $(4 \mid 0)$ 超形式。为简化起见，我们现在假设场满足所有曲率上的洛伦兹水平性约束，因此场实际上仅依赖于超空间坐标 x^μ, θ^α ， 其中满足 $\mu = 1, \ldots 4, \alpha = 1, \ldots 4$ 。此时 $\widetilde{G}$ 就是 $M^{4|4}$ 超空间，只有 (4|4) 型的可积最高次形式可以在 $M^{4|4}$ 上积分。因此我们需要一个 $(0 \mid 4)$ 型的庞加莱对偶，使得

$$L^{4|0} \wedge \eta_{M^4}^{0|4} \tag{144}$$

is an integral top form, i.e., of type (4|4). For this purpose, we choose

成为一个可积最高次形式，即类型为 (4|4)。为此我们选取

$$\eta_{M^4}^{0|4} = \varepsilon_{\alpha\beta\gamma\delta}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta\varepsilon_{\alpha'\beta'\gamma'\delta'}\delta\left(d\theta^{\alpha'}\right) \wedge \delta\left(d\theta^{\beta'}\right) \wedge \delta\left(d\theta^{\gamma'}\right) \wedge \delta\left(d\theta^{\delta'}\right) \tag{145}$$

so that

由此可得

$$\int_{M^{4|4}} L^{4|0} \wedge \eta_{M^4}^{0|4} = \int_{M^4} L^{4|0}\left(\theta = 0, d\theta = 0\right) \tag{146}$$

and we obtain a spacetime action, where all fields depend only on x -coordinates (the terms containing θs are annihilated by the presence of the 4θs in η) and have no "legs" $d\theta$ because of the $\delta(d\theta)$ in η . Note that $\eta_{M^4}^{0|4}$ is closed and the explicit θs prevent it to be exact.

最终我们得到时空作用量，其中所有场都仅依赖于 x 坐标 (含 θs 的项会被 η 中 4θs 的存在湮灭)，并且由于 η 中的 $\delta(d\theta)$ ，场没有“支腿” $d\theta$ 。注意 $\eta_{M^4}^{0|4}$ 是闭形式，而显式的 θs 阻止它成为恰当形式。

Since multiplying by the Poincaré dual changes the picture number of the resulting form, η is also called picture changing operator (PCO), a name borrowed from string theory and string field theory.

由于乘上庞加莱对偶会改变所得形式的图数，η 也被称为图变换算子 (PCO)，这个名称借自弦论和弦场论。

The Poincaré dual is by no means unique: we can orient the M^4 surface inside $\widetilde{G}$ in many different ways. For example, consider the PCO obtained by acting on η with an infinitesimal diffeomorphism in the θ directions:

庞加莱对偶绝非唯一的: 我们可以通过多种不同方式在 $\widetilde{G}$ 内部固定 M^4 曲面的定向。例如，考虑对 η 沿 θ 方向作无穷小微分同胚得到的图变换算子:

$$\eta' = \eta + \ell_\varepsilon \eta = \eta + d\left(\iota_\varepsilon \eta\right) \tag{147}$$

This is still a PCO, being closed and not exact [9] , and dual to a submanifold diffeomorphic to the original M^4 . Note also that the change in η is exact.

它仍然是一个图变换算子: 是闭而非恰当的 [9] ，且对偶于一个与原 M^4 微分同胚的子流形。还需要注意，η 的变化是恰当的。

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[9] Because η is closed and not exact and d commutes with ℓ_ε .

[9] 因为 η 是闭而非恰当的，且 d 与 ℓ_ε 对易。

Gamma Matrices in $d = 3 + 1$

$d = 3 + 1$ 中的伽马矩阵

$$\eta_{ab} = (1, -1, -1, -1)\,,\ \{\gamma_a, \gamma_b\} = 2\eta_{ab},\ [\gamma_a, \gamma_b] = 2\gamma_{ab}, \tag{148}$$

$$\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3,\ \gamma_5\gamma_5 = 1,\ \varepsilon_{0123} = -\varepsilon^{0123} = 1, \tag{149}$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0,\ \gamma_5^\dagger = \gamma_5 \tag{150}$$

$$\gamma_a^T = -C\gamma_a C^{-1},\ \gamma_5^T = C\gamma_5 C^{-1},\ C^2 = -1,\ C^T = -C \tag{151}$$

Useful Identities

有用恒等式

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \tag{152}$$

$$\gamma_{ab}\gamma_5 = -\frac{i}{2}\varepsilon_{abcd}\gamma^{cd} \tag{153}$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + i\varepsilon_{abcd}\gamma_5\gamma^d \tag{154}$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + i\varepsilon_{abcd}\gamma_5\gamma^d \tag{155}$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + i\varepsilon_{abcd}\gamma_5\gamma^d \tag{156}$$

$$\gamma^{ab}\gamma_{cd} = i\varepsilon^{ab}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}_{d]} - 2\delta^{ab}_{cd} \tag{157}$$

Charge Conjugation and Majorana Condition

电荷共轭与马约拉纳条件

$$\text{Dirac conjugate } \overline{\psi} \equiv \psi^\dagger\gamma_0 \tag{158}$$

$$\text{Charge conjugate spinor } \psi^c = C\left(\overline{\psi}\right)^T \tag{159}$$

$$\text{Majorana spinor } \psi^c = \psi \Rightarrow \overline{\psi} = \psi^T C \tag{160}$$

Fierz Identity for Two Spinor One-Forms

两个旋量一形式的菲尔茨恒等式

$$\psi\overline{\chi} = \frac{1}{4}\left[\left(\overline{\chi}\psi\right)1 + \left(\overline{\chi}\gamma_5\psi\right)\gamma_5 + \left(\overline{\chi}\gamma^a\psi\right)\gamma_a + \left(\overline{\chi}\gamma^a\gamma_5\psi\right)\gamma_a\gamma_5 - \frac{1}{2}\left(\overline{\chi}\gamma^{ab}\psi\right)\gamma_{ab}\right]$$

(161)

Fierz Identity for Two Majorana Spinor One-Forms

两个马约拉纳旋量一次型的菲尔茨恒等式

$$\psi\overline{\psi} = \frac{1}{4}\left[\left(\overline{\psi}\gamma^a\psi\right)\gamma_a - \frac{1}{2}\left(\overline{\psi}\gamma^{ab}\psi\right)\gamma_{ab}\right] \tag{162}$$

As a consequence

由此可得

$$\gamma_a \psi \overline{\psi} \gamma^a \psi = 0, \ \psi \overline{\psi} \gamma^a \psi - \gamma_b \psi \overline{\psi} \gamma^{ab} \psi = 0 \tag{163}$$

Cross-References

交叉引用

Covariant Superspace Approaches to $\mathcal{N} = 2$ Supergravity

$\mathcal{N} = 2$ 超引力的协变超空间方法

- 11D Supergravity and Hidden Symmetries
  - 11 维超引力与隐藏对称性
- $\mathcal{N} = 2$ Supergravities in Harmonic Superspace
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Simple Supergravity

简单超引力

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